Applied Cooperative Game Theory

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CO-values and network formation

- we employed a CS-value, the χ-value, to study group formation
- based on some (N, v), we considered group formation games (non-cooperative games in strategic form), which employed the χ-value to determine the players' payoffs
- the coalition structures resulting from strong equilibria of the these games were called χ-stable
- similarly, on one can analyze network formation and stable networks
- several approaches
  - simultaneous link formation—strategic form games
  - sequential link formation—extensive form games

Strategic form games

- $\left(I, (S_i)_{i \in I}, (u_i)_{i \in I}\right)$
- non-empty and finite **player set**: I
- non-empty and finite **sets of (pure) strategies**: S_i, $i \in I$
- typical member: s_i
- set of strategy profiles: $S := \prod_{i \in I} S_i$
- typical member: s
- incomplete strategy profiles for $K \subseteq I$: $S_K := \prod_{i \in K} S_i$
- $S_{-i} := S_{N \setminus \{i\}}$, typical member: s_{-i}
- typical member: s_K
- for $i \in I$, $K \subseteq I$, and $s \in S$, s_i and s_K also denote the obvious restrictions of s
- payoff functions: $u_i : S \to \mathbb{R}$, $i \in I$

Nash equilibrium and undominated Nash equlibrium

- **Nash equilibrium**: $s^* \in S$ such that $u_i(s^*) \geq u_i(s_i s^*_{-i})$ for all $i \in I$ and $s_i \in S_i$
- **domination**: for $i \in N$, $s_i \in S_i$ dominates $s'_i \in S_i$ iff
  - $u_i(s_i s_{-i}) \geq u_i(s'_i s_{-i})$ for all $s_{-i} \in S_{-i}$ and
  - $u_i(s_i s_{-i}) > u_i(s'_i s_{-i})$ for some $s_{-i} \in S_{-i}$.
- **weak domination**: for $i \in N$, $s_i \in S_i$ weakly dominates $s'_i \in S_i$ iff
  - $u_i(s_i s_{-i}) \geq u_i(s'_i s_{-i})$ for all $s_{-i} \in S_{-i}$
- **undominated Nash equilibrium**: $s^* \in S$ such that
  - s^* is a Nash equilibrium,
  - for all $i \in I$, s^*_i is not dominated by some $s_i \in S_i$

Network formation games (NFG) and stable networks

- TU game: (N, v)
- CO-value: μ
- player set: $I = N$
- strategy sets; for $i \in N$, $S_i = \{K \subseteq N | i \in K\}$
- induced network: $L : S \rightarrow 2^{\left(L^N\right)}$, $L(s) := \left\{ij \in L | i \in s_j \wedge j \in s_i\right\}$
- payoff functions: for $i \in N$, $u_i(s) = \mu_i(N, v, L(s))$

- $L \subseteq L^N$ is called **Nash stable** iff there is some Nash equilibrium s^* in the NFG, such that $L(s^*) = L$
- $L \subseteq L^N$ is **undominatedly Nash stable** iff there is some undominated Nash equilibrium s^* in the NFG, such that $L(s^*) = L$

- Dutta, B., van den Nouweland, A., Tijs, S. (1998): Link formation in cooperative situations. International Journal of Game Theory 27: 245–256.
- assumption: in the following, (N, v) is superadditive

Theorem (Dutta, van den Nouweland, and Tijs 1998).
Any network $L \subseteq L^N$ is Nash stable for (N, v).

Proof. Fix $L \subseteq L^N$.

- Consider $s^* \in S$, such that $s_i^* = \{j \in N \setminus \{i\} \mid ij \in L\}, \ i \in N$.
- obviously, this implies $L(s^*) = L$
- further, $L(s_i s_{-i}^*) \subseteq L(s^*)$ for all $i \in N$ and $s_i \in S_i$
- since μ obeys **LM** for superadditive (N, v),

$$u_i(s^*) = \mu_i(N, v, L(s^*)) \geq \mu_i(N, v, L(s_i s^*)) = u_i(s_i s_{-i}^*), \qquad i \in N$$

- hence, s^* is a Nash equilibrium in the NFG $\square$

Theorem (Dutta, van den Nouweland, and Tijs 1998).

The network L^N is undominatedly Nash stable for (N, v).

Proof. (i) If $L(\bar{s}) = L^N$ then $\bar{s}_i = N \setminus \{i\}$, $i \in N$.

- for $s_i \in S_i$, $L(s_i s_{-i}) \subseteq L(\bar{s}_i s_{-i})$, $i \in N$
- hence by **LM**, for all $i \in N$, $s_i \in S_i$, and $s_{-i} \in S_{-i}$

$$u_i(\bar{s}_i s_{-i}) = \mu_i(N, v, L(\bar{s}_i s_{-i})) \geq \mu_i(N, v, L(s_i s_{-i})) = u_i(s_i s_{-i}) \quad (*)$$

- this implies that
- $\bar{s}_i$ is undominated for all $i \in N$
- $\bar{s}$ is a Nash equilibrium □

Theorem (Dutta, van den Nouweland, and Tijs 1998).
If L is undominatedly Nash stable for (N, v), then $\mu(N, v, L) = \mu(N, v, L^N)$.

Lemma (A). If $\mu_k(N, v, L + ij) \neq \mu_k(N, v, L)$, $k \in N \setminus \{i, j\}$, then $\mu_i(N, v, L + ij) > \mu_i(N, v, L)$.

Proof. Obviously, $k \in C_i(N, v, L + ij)$

- If $\mu_i(N, v, L + ij) \leq \mu_i(N, v, L)$, then $\mu_i(N, v, L + ij) = \mu_i(N, v, L)$ by **LM**.
- By **SI**, $\mu_k(N, v, L + ij) < \mu_k(N, v, L)$ and $\mu_\ell(N, v, L + ij) \leq \mu_\ell(N, v, L)$ for all $\ell \in N \setminus \{k\}$.
- Summing up $\mu_\ell(N, v, L + ij) - \mu_\ell(N, v, L)$ over $\ell \in C_i(N, v, L + ij)$ gives

$$v(C_i(N, v, L + ij)) = \mu_{C_i(N,v,L+ij)}(N, v, L + ij)$$
$$< \mu_{C_i(N,v,L+ij)}(N, v, L) = \sum_{C \in \mathcal{C}(N,v,L) : C \subseteq C_i(N,v,L+ij)} v(C)$$

- contradicting the superadditivity of (N, v) □

Lemma (A). Let $s_i, s_i' \in S_i$ and $s_{-i} \in S_{-i}$ such that $s_i \subseteq s_i'$ and $\mu_i(N, v, L(s_i s_{-i})) = \mu_i(N, v, L(s_i' s_{-i}))$, then $\mu(N, v, L(s_i s_{-i})) = \mu(N, v, L(s_i' s_{-i}))$

Proof. If $L(s_i s_{-i}) = L(s_i' s_{-i})$, then the claim obviously holds true.

- else by $s_i \subseteq s_i'$, $L(s_i s_{-i}) \subsetneq L(s_i' s_{-i})$ and $\emptyset \neq L' := L(s_i' s_{-i}) \setminus L(s_i s_{-i}) \subseteq L_i$
- fix $ij \in L'$. applying **LM** repeatedly, we have

$$\mu_i(L(s_i s_{-i})) \leq \mu_i(L(s_i s_{-i}) + ij) \leq \mu_i\left(L\left(s_i' s_{-i}\right)\right)$$

- by $\mu_i(L(s_i s_{-i})) = \mu_i(L(s_i' s_{-i}))$, we have $\mu_i(L(s_i s_{-i})) = \mu_i(L(s_i s_{-i}) + ij)$
- by **F,** we have $\mu_j(L(s_i s_{-i})) = \mu_j(L(s_i s_{-i}) + ij)$
- by Lemma (A), $\mu_k(L(s_i s_{-i})) = \mu_k(L(s_i s_{-i}) + ij)$, $k \in N \setminus \{i, j\}$
- hence, $\mu(L(s_i s_{-i})) = \mu(L(s_i s_{-i}) + ij)$
- adding all links from L' finally shows $\mu(L(s_i s_{-i})) = \mu(L(s_i' s_{-i}))$ □

Proof. (Theorem)

- let $s \neq \bar{s}$ be an undominated Nash equilibrium of the NFG
- since any s_i is undominated and by (*),

$$\mu_i\left(L\left(\bar{s}_i s_{-i}\right)\right) = \mu_i\left(L\left(s_i s_{-i}\right)\right), \qquad s_{-i} \in S_{-i}$$

- in particular, for any $i \in N = \{1, \ldots, n\}$

$$\mu_i\left(L\left(\bar{s}_{\{1,\ldots,i\}} s_{\{i+1,\ldots,n\}}\right)\right) = \mu_i\left(L\left(\bar{s}_{\{1,\ldots,i-1\}} s_{\{i,\ldots,n\}}\right)\right)$$

- since $s_i \subseteq \bar{s}_i$, by Lemma B,

$$\mu\left(L\left(\bar{s}_{\{1,\ldots,i\}} s_{\{i+1,\ldots,n\}}\right)\right) = \mu\left(L\left(\bar{s}_{\{1,\ldots,i-1\}} s_{\{i,\ldots,n\}}\right)\right)$$

- entailing $\mu\left(L\left(\bar{s}\right)\right) = \mu\left(L\left(s\right)\right)$ □

Definition. A **potential** P for CS-games is an operator that assigns to any CS-game $(N, v, \mathcal{P})$ a number $P(N, v, \mathcal{P}) \in \mathbb{R}$ such that
(i) $P(\emptyset, v, \mathcal{P}) = 0$,
(ii) $\sum_{i\in N} \left[P(N, v, \mathcal{P}) - P\left(N\backslash\{i\}, v|_{N\backslash\{i\}}, \mathcal{P}|_{N\backslash\{i\}}\right)\right] = \sum_{C\in\mathcal{P}} v(C)$.

Theorem. There is a unique potential for CS-games, which satisfies $P(N, v, \mathcal{P}) - P\left(N\backslash\{i\}, v|_{N\backslash\{i\}}, \mathcal{P}|_{N\backslash\{i\}}\right) = \mathrm{AD}_i(N, v, \mathcal{P})$, $i \in N$.

Proof. Uniqueness: Let P, Q be two potentials. We show $P = Q$.

- *Induction basis:* $|N| = 1$. by (i+ii), $P(\{i\}, v, \mathcal{P}) = Q(\{i\}, v, \mathcal{P}) = v(\{i\})$.
- *Induction hypothesis (H)*: $P = Q$ for $|N| \leq k$
- *Induction step*: let $|N| = k + 1$. this implies,

$$P(N, v, \mathcal{P}) \overset{\text{(ii)}}{=} \frac{\sum_{C\in\mathcal{P}} v(C) + \sum\limits_{i\in N} P\left(N\backslash\{i\}, v|_{N\backslash\{i\}}, \mathcal{P}|_{N\backslash\{i\}}\right)}{|N|} \overset{\text{H}}{=} Q(N, v, \mathcal{P})$$

- existence: consider the operator P given by

$$P\left(N, v, \mathcal{P}\right) := \sum_{C \in \mathcal{P}} \sum_{T \subseteq C, T \neq \emptyset} \frac{\lambda_T\left(v\right)}{|T|}$$

- this gives $P\left(\emptyset, v, \mathcal{P}\right) = 0$ and

$$\begin{aligned} & \sum_{i \in N} P\left(N, v, \mathcal{P}\right) - P\left(N \backslash \{i\}, v|_{N \backslash \{i\}}, \mathcal{P}|_{N \backslash \{i\}}\right) \\ = \; & \sum_{i \in N} \left[\sum_{C \in \mathcal{P}} \sum_{T \subseteq C, T \neq \emptyset} \frac{\lambda_T\left(v\right)}{|T|} - \sum_{C \in \mathcal{P}|_{N \backslash \{i\}}} \sum_{T \subseteq C, T \neq \emptyset} \frac{\lambda_T\left(v\right)}{|T|} \right] \end{aligned}$$

The potential approach to the AD value #3

- since removing i only affects the component $\mathcal{P}(i)$, we obtain

$$\begin{aligned}
&= \sum_{i\in N}\left[\sum_{T\subseteq\mathcal{P}(i),T\neq\emptyset}\frac{\lambda_T(v)}{|T|}-\sum_{T\subseteq\mathcal{P}(i)\setminus\{i\},T\neq\emptyset}\frac{\lambda_T(v)}{|T|}\right]\\
&= \sum_{i\in N}\left[\sum_{T\subseteq\mathcal{P}(i),i\in T}\frac{\lambda_T(v)}{|T|}\right]\\
&= \sum_{i\in N}\mathrm{Sh}_i\left(\mathcal{P}(i),v|_{\mathcal{P}(i)}\right)\\
&= \sum_{i\in N}\mathrm{AD}_i(N,v,\mathcal{P})\\
&= \sum_{C\in\mathcal{P}}v(P)
\end{aligned}$$

- a **probabilistic coalition structure** on (N, v) is a probability distribution p on the set $\mathbb{P}(N)$ of all coalition structures on N
- let $\Delta(\mathbb{P}(N))$ denote the set of these probability distributions
- a **probabilistic CS-game** (pCS-game) (N, v, p) is a TU game (N, v) together with a probabilistic coalition structure $p \in \Delta(\mathbb{P}(N))$
- a **probabilistic CS-value** is an operator φ that assigns to any pCS-game (N, v, p) some payoff vector $\varphi(N, v, p) \in \mathbb{R}^N$
- $\mathbb{P}(N)$ can be viewed as subset of $\Delta(\mathbb{P}(N))$ in a canonical way: $\mathcal{P} \equiv p_{\mathcal{P}} \in \Delta(\mathbb{P}(N))$, $p_{\mathcal{P}}(\mathcal{P}) = 1$
- the AD-value and the χ-value can be extended to pCS-games in a natural way

$$\begin{aligned} \mathrm{AD}(N, v, p) &= \sum_{\mathcal{P} \in \mathbb{P}(N)} p(\mathcal{P})\, \mathrm{AD}(N, v, \mathcal{P}) \\ \chi(N, v, p) &= \sum_{\mathcal{P} \in \mathbb{P}(N)} p(\mathcal{P})\, \chi(N, v, \mathcal{P}) \end{aligned}$$

Problem. There is an easy way to characterize these pCS-values using axiomatizations of the underlying CS-values. Try to find it!