Applied Cooperative Game Theory

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Communication and bilateral contracts

- coalition structures (partitions of the player set) are a rather coarse way to model restricted cooperation
- possibility of cooperation may depend on
  - communication between players
  - bilateral contracts
- example: gloves game; one right-glove holder, r, and one left-glove holder, ℓ, actually sell their pair of gloves which is worth 1 via some agent, $A1$, who is necessary do facilitate the deal
- therefore, the agent a should obtain some share of the proceeds of 1
- how to model this? TU game (N, v), $N = \{r, \ell, a\}$, $v(K) = 1$ if $\{r, \ell\} \subseteq K$, else $v(K) = 0$
- coalition structure: $\mathcal{P} = \{N\}$? inadequate, because this does not reflect the fact that r and ℓ need a as a sales agent
- indeed: $\mathrm{AD}_a(N, v, \mathcal{P}) = \chi_a(N, v, \mathcal{P}) = 0$
- instead of $\mathcal{P}$ consider the (undirected) graph

  L: $r \bullet$ —— $a \bullet$ —— $\ell \bullet$

- Myerson value: $\mu_a(N, v, L) = \frac{1}{3}$; all players are necessary to generate the worth of 1

- an **undirected graph** is a pair (N, L)
  - non-empty and finite set N
  - set of **links** L : $L \subseteq L^N := \{\{i,j\} \,|\, i,j \in N,\ i \neq j\}$
  - typical element of L^N: λ or $ij := \{i,j\}$
- $i,j \in N$, $i \neq j$ are **directly connected** in (N, L) iff $ij \in L$
- $i,j \in N$, $i \neq j$ are **connected** in (N, L) iff there is a finite sequence of players $(i_1, \ldots, i_n)$ such that $\{i_k, i_{k+1}\} \in L$, $k = 1, \ldots, n-1$
- the binary relation "connected with" is reflexive, symmetric, and transitive, i.e., an equivalence relation which induces equivalence classes $C \subseteq N$: $i,j \in C$ iff i and j are connected in (N, L)
- these equivalence classes are called the **(connected) components of** (N, L); $C_i(N, L)$ stands for the connected component containing player i
- so, any graph (N, L) induces a partition of N, the set of the connected components: $\mathcal{C}(N, L) := \{C_i(N, L) \,|\, i \in N\}$
- for any graph (N, L) and $K \subseteq N$, $L|_K$ denotes the restriction of L to K:

$$L|_K := L \cap L^K = \{\lambda \in L | \lambda \subseteq K\}$$

- (N, L) is called a **cooperation structure** (on N)

Some facts on connected components

- if $L' \subseteq L \subseteq L^K$, $K \subseteq N$, then $\mathcal{C}(K, L')$ is finer than $\mathcal{C}(K, L)$
- if $K' \subseteq K \subseteq N$, then for any $C' \in \mathcal{C}(K', L|_{K'})$ there is some $C \in \mathcal{C}(K, L|_K)$ such that $C' \subseteq C$
- if $i \notin K \subseteq N$, $L \subseteq L^N$, then $\mathcal{C}(K, L|_K) = \mathcal{C}(K, L - ij|_K)$
- $S, T \subseteq N$, $S \cap T = \emptyset$, $L' \subseteq L^S$, $L \subseteq L^T$: $\mathcal{C}(S \cup T, L' \cup L) = \mathcal{C}(S, L') \cup \mathcal{C}(T, L)$
- $S, T \subseteq N$, $L \subseteq L^N$: $L|_S \cup L|_T \subseteq L|_{S \cup T}$

- a TU game (N, v) together with an undirected graph (N, L) is called a **TU game with a cooperation structure** or a **CO-game**, for short
- a **solution for CO-games (CO-solution, CO-value)** is an operator φ that assigns payoffs $\varphi(N, v, L) \in \mathbb{R}^N$ to any CO-game (N, v, L)
- of course, any CS-solution φ gives rise to a CO-solution φ^{CO} via

$$\varphi^{CO}(N, v, L) := \varphi(N, v, \mathcal{C}(N, L))$$

- the other way round, any CO-solution φ gives rise to a CS-solution φ^{CS} via

$$\varphi^{CS}(N, v, \mathcal{P}) := \varphi\left(N, v, L^{\mathcal{P}}\right)$$

  where

$$L^{\mathcal{P}} := \bigcup_{C \in \mathcal{C}(N,L)} L^C$$

- in $L^{\mathcal{P}}$ the components of $\mathcal{P}$ are internally completely connected by links, but there are no links between components; obviously, $\mathcal{C}\left(N, L^{\mathcal{P}}\right) = \mathcal{P}$
- a CO-solution ψ generalizes CS-solution φ if $\psi\left(N, v, L^{\mathcal{P}}\right) = \varphi(N, v, \mathcal{P})$

- for a coalition function $v \in \mathbb{V}(N)$ and a graph (N, L), we define the **graph restricted coalition function** $v^L \in \mathbb{V}(N)$ as follows:

$$v^L(K) := \sum_{S \in \mathcal{C}(K, L|_K)} v(S), \qquad K \subseteq N$$

- looks more difficult than it is
- what is $\mathcal{C}(K, L|_K)$? well, the set of components of K which are connected **within** K
- interpretation: players in K are only able to cooperate to create worth when they are connected in K

- obviously, $K \subseteq N$: $\mathcal{C}\left(K, L^N|_K\right) = \mathcal{C}\left(K, L^K\right) = \{K\}$, hence, $v^{L^N} = v$
- moreover, $P \in \mathcal{P}$, $K \subseteq P$: $L^{\mathcal{P}}|_K = L^K$; $\mathcal{C}\left(K, L^{\mathcal{P}}|_K\right) = \mathcal{C}\left(K, L^K\right) = \{K\}$
- $v \in \mathbb{V}(N)$: $v^{L^{\mathcal{P}}}|_P = v|_P \in \mathbb{V}(P)$; $v^{L^{\mathcal{P}}|_P} = v^{L^P} \in \mathbb{V}(P)$

- question: which properties of $v \in \mathbb{V}(N)$ are inherited by v^L?
- look at: monotonicity, superadditivity, and convexity

Lemma. If $v \in N$ is superadditive, then v^L is superadditive for any $L \subseteq L^N$.

- **Proof.** let $v \in \mathbb{V}(N)$ be superadditive, and $L \subseteq L^N$
- let $S, T \subseteq N$, $S \cap T = \emptyset$; to show: $v^L(S \cup T) \geq v^L(S) + v^L(T)$

$$\begin{aligned} v^L(S) + v^L(T) &= \sum_{K \in \mathcal{C}(S, L|_S)} v(K) + \sum_{K \in \mathcal{C}(T, L|_T)} v(K) \\ &= \sum_{K \in \mathcal{C}(S \cup T, L|_S \cup L|_T)} v(K) \\ &\leq \sum_{K \in \mathcal{C}(S \cup T, L|_{S \cup T})} v(K) = v^L(S \cup T) \end{aligned}$$

- the second equality drops from $S \cap T = \emptyset$ and the construction of $L|_K$
- the inequality drops from v being superadditive and the fact that $\mathcal{C}(S \cup T, L|_S \cup L|_T)$ is finer than $\mathcal{C}(S \cup T, L|_{S \cup T})$:
- $L|_S \cup L|_T \subseteq L|_{S \cup T}$, hence, all players who are connect with each other for $L|_S \cup L|_T$ are connected in $L|_{S \cup T}$ $\square$

- monotonicity and convexity are not inherited, in general
- example: $N = \{1, 2, 3\}$, $v(K) = 1$ if $|K| \geq 1$, else $v(K) = 0$, $L = \{\{1, 2\}, \{2, 3\}\}$; obviously, (N, v) is monotonic
  - however, $\mathcal{C}\left(\{1, 3\}, L|_{\{1,3\}}\right) = \mathcal{C}(\{1, 3\}, \emptyset) = \{\{1\}, \{3\}\}$; $v^L(\{1, 3\}) = v(\{1\}) + v(\{3\}) = 1 + 1 = 2$
  - but, $\mathcal{C}\left(\{1, 2, 3\}, L|_{\{1,2,3\}}\right) = \mathcal{C}(\{1, 2, 3\}, L) = \{\{1, 2, 3\}\}$; $v^L(\{1, 2, 3\}) = v(\{1, 2, 3\}) = 1 < 2$; hence, $\left(N, v^L\right)$ is not monotonic
- example: $N = \{1, 2, 3, 4\}$, $v(K) = |K|^2$ if $|K| > 1$, else $v(K) = 0$, $L = \{\{1, 2\}, \{1, 3\}, \{4, 2\}, \{4, 3\}\}$
  - easy to check that (N, v) is convex $\equiv$ non-decreasing marginal contributions
  - $\mathcal{C}\left(\{2, 3\}, L|_{\{2,3\}}\right) = \mathcal{C}(\{2, 3\}, \emptyset) = \{\{2\}, \{3\}\}$; $v^L(\{2, 3\}) = v(\{2\}) + v(\{3\}) = 0 + 0 = 0$
  - $\mathcal{C}\left(\{1, 2, 3\}, L|_{\{1,2,3\}}\right) = \mathcal{C}(\{1, 2, 3\}, \{\{1, 2\}, \{1, 3\}\}) = \{\{1, 2, 3\}\}$; $v^L(\{1, 2, 3\}) = v(\{1, 2, 3\}) = 3^2 = 9$; analogously, $v^L(\{2, 3, 4\}) = 9$
  - $\mathcal{C}(N, L) = \{N\}$; $v^L(\{1, 2, 3, 4\}) = v(\{1, 2, 3, 4\}) = 4^2 = 16$
  - so, $MC_1^{v^L}(\{2, 3\}) = 9 - 0 > 16 - 9 = MC_1^{v^L}(\{2, 3, 4\})$
  - hence, $\left(N, v^L\right)$ is not convex

- Myerson R. B. (1977) Graphs and cooperation in games. Mathematics of Operations Research 2:225–229

Definition. The Myerson value assigns to any CO-game (N, v, L) and $i \in N$ the payoff

$$\mu_i(N, v, L) := \mathrm{Sh}_i(N, v^L).$$

- simply the Shapley value applied to the graph restricted coalition function
- for $L = L^N$, $v^{L^N} = v$, hence, $\mu\left(N, v, L^N\right) = \mathrm{Sh}\left(N, v\right)$, i.e., μ generalizes Sh
- $v \in \mathbb{V}\left(N\right)$: for $L = L^{\mathcal{P}}$, $P \in \mathcal{P}$, $v^{L^{\mathcal{P}}}|_P = v|_P^{L^{\mathcal{P}}|_P} = v|_P \in \mathbb{V}\left(P\right)$, hence,

$$\begin{aligned}\mu_i\left(N, v, L^{\mathcal{P}}\right) &= \mu_i\left(P, v|_P, L^{\mathcal{P}}|_P\right) = \mathrm{Sh}_i(P, v|_P^{L^{\mathcal{P}}|_P}) \\ &= \mathrm{Sh}_i\left(P, v|_P\right) = \mathrm{AD}_i\left(N, v, \mathcal{P}\right),\end{aligned}$$

- i.e., μ generalizes AD; of course, the first equation has to be shown

Component decomposability, CD For all $i \in C \in \mathcal{C}(N, L)$,

$$\varphi_i(N, v, L) = \varphi_i(C, v|_C, L|_C).$$

Proposition μ satisfies **CD**.
Proof. see literatur

Component efficiency, CE For all $C \in \mathcal{C}(N, L)$, $\varphi_C(N, v, L) = v(C)$.

Proposition μ satisfies **CE**.

Proof.

- since μ meets **CD**, it suffices to show $\mu_C(C, v|_C, L|_C) = v(C)$ for all $C \in \mathcal{C}(N, L)$

$$\begin{aligned}
\sum_{i \in C} \mu_i(C, v|_C, L|_C) &= \sum_{i \in C} \frac{1}{|\Sigma(C)|} \sum_{\rho \in \Sigma(C)} MC_i\left(\rho, v|_C^{L|_C}\right) \\
&= \frac{1}{|\Sigma(C)|} \sum_{\rho \in \Sigma(C)} \sum_{i \in C} MC_i\left(\rho, v|_C^{L|_C}\right) \\
&= \frac{1}{|\Sigma(C)|} \sum_{\rho \in \Sigma(C)} \left(v|_C^{L|_C}(C) - v|_C^{L|_C}(\emptyset)\right) \\
&= \frac{1}{|\Sigma(C)|} \sum_{\rho \in \Sigma(C)} v|_C(C) \\
&= \frac{1}{|\Sigma(C)|} \sum_{\rho \in \Sigma(C)} v(C) = v(C)
\end{aligned}$$

□

Fairness, F For all $ij \in L$, we have

$$\varphi_i(N, v, L) - \varphi_i(N, v, L - ij) = \varphi_j(N, v, L) - \varphi_j(N, v, L - ij).$$

Proposition μ satisfies **F**.
Proof.

- let $\sigma, \rho \in \Sigma(N)$, $\sigma(i) = \rho(j) > \sigma(j) = \rho(i)$, and $\sigma(\ell) = \rho(\ell)$ for $\ell \in N \backslash \{i, j\}$
- by definition of μ it suffices to show

$$\begin{aligned} & MC_i\left(\sigma, v^L\right) - MC_i\left(\sigma, v^{L-ij}\right) + MC_i\left(\rho, v^L\right) - MC_i\left(\rho, v^{L-ij}\right) \\ = \; & MC_j\left(\sigma, v^L\right) - MC_j\left(\sigma, v^{L-ij}\right) + MC_j\left(\rho, v^L\right) - MC_j\left(\rho, v^{L-ij}\right) \end{aligned}$$

- since $K_i(\sigma) = K_j(\rho)$, $j \notin K_i(\rho)$, and $i \notin K_j(\sigma)$, we have

$$\begin{aligned}
& MC_i\left(\sigma, v^L\right) - MC_i\left(\sigma, v^{L-ij}\right) + MC_i\left(\rho, v^L\right) - MC_i\left(\rho, v^{L-ij}\right) \\
= \quad & \left[v^L\left(K_i(\sigma)\right) - v^L\left(K_i(\sigma)\setminus i\right)\right] - \left[v^{L-ij}\left(K_i(\sigma)\right) - v^{L-ij}\left(K_i(\sigma)\setminus i\right)\right] + \\
= \quad & v^L\left(K_i(\sigma)\right) - v^{L-ij}\left(K_i(\sigma)\right) + v^L\left(K_i(\rho)\right) - v^{L-ij}\left(K_i(\rho)\right) \\
= \quad & v^L\left(K_i(\sigma)\right) - v^{L-ij}\left(K_i(\sigma)\right) + 0 \\
= \quad & v^L\left(K_j(\rho)\right) - v^{L-ij}\left(K_j(\rho)\right) + 0 \\
= \quad & v^L\left(K_j(\rho)\right) - v^{L-ij}\left(K_j(\rho)\right) + v^L\left(K_j(\sigma)\right) - v^{L-ij}\left(K_j(\sigma)\right) \\
= \quad & MC_j\left(\sigma, v^L\right) - MC_j\left(\sigma, v^{L-ij}\right) + MC_j\left(\rho, v^L\right) - MC_j\left(\rho, v^{L-ij}\right)
\end{aligned}$$

$\square$

Link monotonicity, LM. For all $i, j \in N$, $\varphi_i(N, v, L + ij) \geq \varphi_i(N, v, L)$.

Proposition. μ satisfies **LM** for superadditive games.

- drops from the next proposition
- suppose $\varphi_i(N, v, L + ij) - \varphi_i(N, v, L) < 0$ for some $i, j \in N$; of course, $ij \notin L$
- since μ meets **SI**, for all $k \in \mathcal{C}_i(N, L + ij)$,
$$\mu_k(N, v, L + ij) < \mu_k(N, v, L)$$
- summing up over $k \in C_i(N, L + ij)$ gives
$$\mu_{C_i(N,L+ij)}(N, v, L + ij) < \mu_{C_i(N,L+ij)}(N, v, L)$$
- obviously, $\mathcal{C}(N, L + ij)$ coarser than $\mathcal{C}(N, L)$, hence
$$C_i(N, L + ij) = \bigcup_{C \in \mathcal{C}(N,L): C \subseteq C_i(N,L+ij)} C$$
- since μ satisfies **CE**, we have
$$v(C_i(N, L + ij)) < \sum_{C \in \mathcal{C}(N,L): C \subseteq C_i(N,L+ij)} v(C),$$
contradicting, superadditivity of v □

Strong improvement, SI. For all $i, j, k \in N$,
$\varphi_i(N, v, L + ij) - \varphi_i(N, v, L) \geq \varphi_k(N, v, L + ij) - \varphi_k(N, v, L)$.

Proposition. μ satisfies **SI** for superadditive games.

- let (N, v) be superadditive; let $i, j, k \in N$
- let $\sigma, \rho \in \Sigma(N)$, $\sigma(i) = \rho(k) > \sigma(k) = \sigma(i)$, and $\sigma(\ell) = \rho(\ell)$ for $\ell \in N \backslash \{i, k\}$
- by definition of v^L and the superadditivity of (N, v), we have

$$MC_i\left(\rho, v^{L+ij}\right) - MC_i\left(\rho, v^L\right) \geq 0 = MC_k\left(\sigma, v^{L+ij}\right) - MC_k\left(\sigma, v^L\right) \quad (*)$$

- further,

$$MC_i\left(\sigma, v^{L+ij}\right) - MC_i\left(\sigma, v^L\right) = v^{L+ij}(K_i(\sigma)) - v^L(K_i(\sigma)),$$

  because $i \notin S$ implies $v^{L+ij}(S) = v^L(S)$
- hence by $K_i(\sigma) = K_k(\rho)$, we have

$$\begin{aligned} MC_k\left(\rho, v^{L+ij}\right) - MC_k\left(\rho, v^L\right) &= MC_i\left(\sigma, v^{L+ij}\right) - MC_i\left(\sigma, v^L\right) \\ &\quad + v^L(K_i(\sigma) \backslash k) - v^{L+ij}(K_i(\sigma) \backslash k) \end{aligned}$$

Strong improvement #2

- since $\mathcal{C}\left(K_i(\sigma)\setminus k, L|_{K_i(\sigma)\setminus k}\right)$ is finer than $\mathcal{C}\left(K_i(\sigma)\setminus k, L+ij|_{K_i(\sigma)\setminus k}\right)$, the superadditivity of (N, v) and definition of v^L imply

$$v^L(K_i(\sigma)\setminus k) \leq v^{L+ij}(K_i(\sigma)\setminus k),$$

hence,

$$MC_i\left(\sigma, v^{L+ij}\right) - MC_i\left(\sigma, v^L\right) \geq MC_k\left(\rho, v^{L+ij}\right) - MC_k\left(\rho, v^L\right) \quad (**)$$

- by definition of μ, (*) and (**) together prove the claim □

Theorem (Myerson 1977). The Myerson value is the unique value that satisfies **CE** and **F**.

Proof. already shown: μ obeys **CE** and **F**

- let φ and ψ satisfy **CE** and **F**, but $\varphi \neq \psi$
- let $L \subseteq L^N$ be some smallest link set such that $\varphi(N, v, L) \neq \psi(N, v, L)$
- by **CE**, $L \neq \emptyset$; because $\varphi_i(N, v, L) = v(\{i\}) = \psi_i(N, v, L)$ if $C_i(N, L) = \{i\}$
- for $ij \in L$, by **F**;

$$\begin{aligned} \varphi_i(N, v, L) - \varphi_j(N, v, L) &= \varphi_i(N, v, L - ij) - \varphi_j(N, v, L - ij) \\ &= \psi_i(N, v, L - ij) - \psi_j(N, v, L - ij) \\ &= \psi_i(N, v, L) - \psi_j(N, v, L) \end{aligned}$$

i.e.,

$$\varphi_i(N, v, L) - \psi_i(N, v, L) = \varphi_j(N, v, L) - \psi_j(N, v, L)$$

- hence, for all $j \in C_i(N, L)$,

$$\varphi_i(N, v, L) - \psi_i(N, v, L) = \varphi_j(N, v, L) - \psi_j(N, v, L)$$

- summing up over $j \in C_i(N, L)$, we have

$$\begin{aligned} &|C_i(N, L)| (\varphi_i(N, v, L) - \psi_i(N, v, L)) \\ &= \varphi_{C_i(N,L)}(N, v, L) - \psi_{C_i(N,L)}(N, v, L) \\ &= v(C_i(N, L)) - v(C_i(N, L)) \\ &= 0 \end{aligned}$$

- hence, $\varphi_i(N, v, L) = \psi_i(N, v, L)$, contradiction □

Alternative proof.

- let φ obey **CE** and **F**; we show $\varphi = \mu$ by induction on $|L|$
- *Induction basis*: by **CE**, the claim holds for $|L| = 0$
- *Induction hypothesis (H)*: the claim holds for $|L| = k$
- *Induction step*: let $|L| = k + 1$
- for $|C_i(N, L)| = 1$, the claim follows from **CE**
- fix $C \in \mathcal{C}(N, L)$, $|C| > 1$
- note: $|L|_C| \geq |C| - 1$ because $(C, L|_C)$ is connected
- by **F**, $(\varphi_i(N, v, L))_{i \in C}$ satisfies the following system of linear equations: for $ij \in L|_C$

$$\begin{aligned} \varphi_i(N, v, L) - \varphi_j(N, v, L) &= \varphi_i(N, v, L - ij) - \varphi_j(N, v, L - ij) \\ &\overset{\mathbf{H}}{=} \mu_i(N, v, L - ij) - \mu_j(N, v, L - ij) \\ \sum_{i \in C} \varphi_i(N, v, L) &\overset{\mathbf{CE}}{=} v(C) \end{aligned}$$

- from the coefficient structure it is clear, that this system has at most one solution
- since the μ-payoffs satisfy these equations, we have $\varphi = \mu$