Problem 1 (7 points)
A producer faces the following production function:

$$y = f(x_1, x_2) = \min\{a \cdot x_1, x_2\}$$

where y is the output and x_1 and x_2 denote the input factors, $a > 0$.
Assume prices $w_1 = 1$ and $w_2 = 2$.

a) Determine the cost function!

Now assume that the parameter a is uncertain. With probability q, we have $a = a_\ell = 3$, while with probability $1-q$, $a = a_h = 4$. The firm determines the inputs after learning the realization of a.

b) Determine the cost lottery (with costs rather than payoffs) if 12 units of output should be produced.

Solution:

a) On optimal relation of the input factors is achieved if $ax_1 = x_2$. Then we have $y = ax_1 = x_2$ what gives

$$\begin{aligned} x_1(y) &= \frac{y}{a}, \\ x_2(y) &= y. \end{aligned}$$

Including this information into

$$c(y) = w_1 \cdot x_1(y) + w_2 \cdot x_2(y)$$

yields

$$c(y) = w_1 \cdot \frac{y}{a} + w_2 \cdot y.$$

Finally, we get

$$c(y) = \left(\frac{1}{a} + 2\right) \cdot y.$$

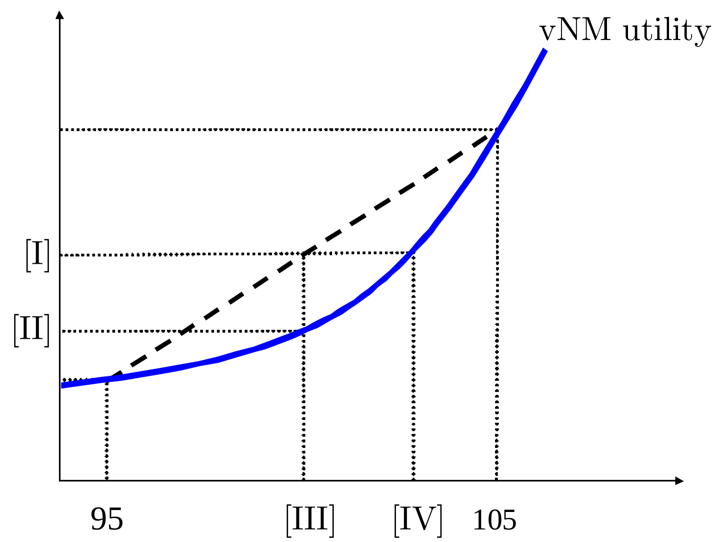
b) If parameter a is high, the costs are $c_h(12) = \left(\frac{1}{a_h} + 2\right) \cdot 12$. Otherwise, costs $c_\ell(12) = \left(\frac{1}{a_\ell} + 2\right) \cdot 12$ occur. Thus, the lottery is given by

$$L = [28, 27; q, 1 - q].$$

Problem 2 (4 points)
Consider two "goods" with points $(-2,2)$, $(-1,2)$ and $(-3,-2)$ in a production set. Focusing on these input-output vectors, only, specify all efficiency relations that hold, or do not hold, between these points!

Solution:
$(-2,2)$ is not efficient since $(-1,2)$ is an improvement upon $(-2,2)$ since $(-1,2) > (-2,2)$. Both $(-2,2)$ and $(-1,2)$ are improvements upon $(-3,-2)$ because $(-2,2) > (-3,-2)$ and $(-1,2) > (-3,-2)$. (Indeed, we even have $(-2,2) \gg (-3,-2)$ and $(-1,2) \gg (-3,-2)$.) (Thus, only $(-1,2)$ is efficient.)

Problem 3 (10 points)
Consider the following figure in order to analyze the lottery $\left[95, 105; \frac{1}{2}, \frac{1}{2}\right]$:



(a) Right or wrong?

	true	false
The depicted vNM utility function exhibits risk aversion.		
The vNM utility function $u(x) = \sqrt{x}$ exhibits risk aversion.		
The vNM utility function $u(x) = 2x^2 + 4$ exhibits risk neutrality.		

(b) Assign I, II, III, and IV in the above figure to the concepts in the table!

	$CE(L)$	$E(L)$	$u(105)$	$u(95)$	$u(E(L))$	$RP(L)$	$E_u(L)$
[I]							
[II]							
[III]							
[IV]							

(c) Assume the following vNM-utility function:

$$u(x) = ax^2 + c, \ a > 0$$

Determine the certainty equivalent of the lottery $\left[2, 3; \frac{4}{5}, \frac{1}{5}\right]$!

Solution:

(a) Right or wrong?

	true	false
The depicted vNM utility function exhibits risk aversion.		$\times$
The vNM utility function $u(x) = \sqrt{x}$ exhibits risk aversion.	$\times$	
The vNM utility function $u(x) = 2x^2 + 4$ exhibits risk neutrality.		$\times$

(b) Assign I, II, III, and IV in the above figure to the concepts in the table!

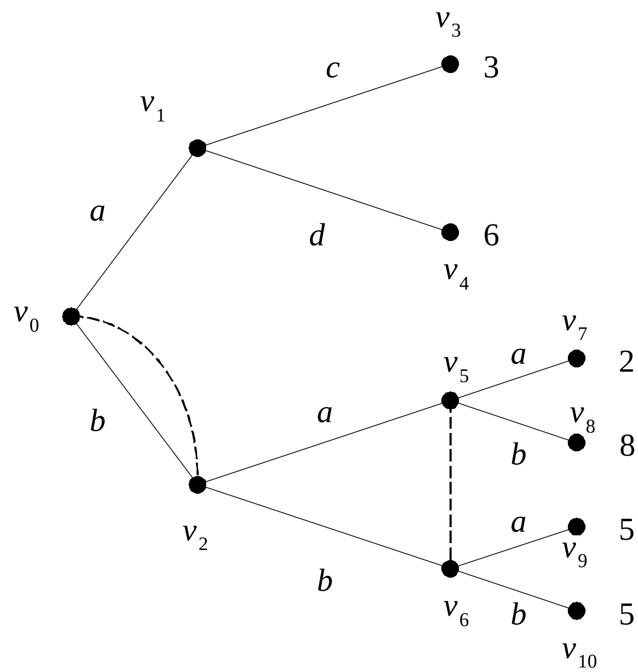
	$CE\left(L\right)$	$E\left(L\right)$	$u\left(105\right)$	$u\left(95\right)$	$u\left(E\left(L\right)\right)$	$RP\left(L\right)$	$E_u\left(L\right)$
[I]							$\times$
[II]					$\times$		
[III]		$\times$					
[IV]	$\times$						

(c) We may use v, given by $v\left(x\right) = x^2$, instead of u, since both are equivalent. From $v\left(CE\right) = E_v\left(L\right)$ and

$$E_v\left(L\right) = \frac{4}{5} \cdot 2^2 + \frac{1}{5} \cdot 3^2 = 5$$

we infer $CE^2 = v\left(CE\right) = 5$, hence $CE = \sqrt{5}$.

Problem 4 (12 points)
Consider the following decision problem without moves by nature!



a) How many subtrees does this decision tree have? Give their initial nodes!

b) Show that this decision situation exhibits imperfect recall!

c) How many strategies can you find? Give two examples.

d) Find the optimal strategies!

Solution:

a) There are two subtrees starting at v_0 and v_1.

b) $I(v_0) = I(v_2)$, but $X(v_0) = (\{v_0, v_2\}) \neq (\{v_0, v_2\}, b, \{v_0, v_2\}) = X(v_2)$, which implies imperfect recall.

c) We have 2^3 strategies (actions for the information sets $\{v_0, v_2\}, \{v_1\}$, and $\{v_5, v_6\}$), for example $[a, c, a]$ and $[a, c, b]$.

d) The payoff 8 cannot be reached by pure strategies. Hence the optimal strategies yield the payoff 6. These are $[a, d, a]$ and $[a, d, b]$.

Problem 5 (10 points)
Consider the following two person game! Calculate all equilibria in pure and properly mixed strategies! Illustrate both reaction functions graphically!

		player 2	
		A	B
player 1	C	$4,2$	$3,3$
	D	$0,4$	$4,5$

Solution:

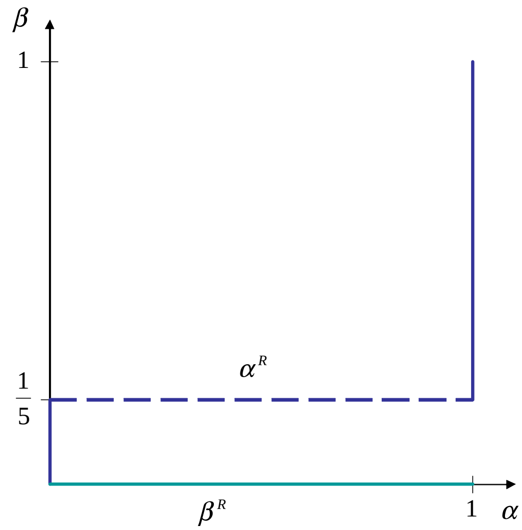
If player 1 chooses strategy C with probability α and player 2 chooses with probability β the expected utility functions of the players are given by

$$\begin{aligned} u_1(\alpha,\beta) &= 4\alpha\beta + 3\alpha(1-\beta) + 4(1-\alpha)(1-\beta) \\ u_2(\alpha,\beta) &= 2\alpha\beta + 3\alpha(1-\beta) + 4(1-\alpha)\beta + 5(1-\alpha)(1-\beta) \end{aligned}$$

The reaction functions can be derived by

$$\begin{aligned} \frac{\partial u_1(\alpha,\beta)}{\partial\alpha} &= 4\beta + 3(1-\beta) - 4(1-\beta) \\ &= -1 + 5\beta \\ \alpha^R(\beta) &= \begin{cases} 0 & \beta < \frac{1}{5} \\ [0,1] & \beta = \frac{1}{5} \\ 1 & \beta > \frac{1}{5} \end{cases} \\ \frac{\partial u_2(\alpha,\beta)}{\partial\beta} &= 2\alpha - 3\alpha + 4(1-\alpha) - 5(1-\alpha) \\ &= -1 \\ \beta^R(\alpha) &= 0, \alpha \in [0,1] \end{aligned}$$

The graphs for the reaction functions are:



We can find one intersection of the reaction functions and therefore there is only one equilibrium (in pure strategies (D, B)) and no other equilibrium in properly mixed strategies.

Problem 6 (13 points)
Consider the utility function

$$U(x_1, x_2) = x_1^2 + 4x_2^2!$$

Assume $p_1, p_2 > 0$, $2p_1 < p_2$ and denote m as the monetary income of the household.

a) Calculate the marginal rate of substitution! Does this utility function represent convex preferences?

b) Determine the indirect utility function for U!

c) Derive the Hicksian demand!

Solution

a) The marginal rate of substitution is given by:

$$MRS = \frac{\frac{\partial U}{\partial x_1}}{\frac{\partial U}{\partial x_2}} = \frac{x_1}{4x_2}.$$

We see that the MRS increases if x_1 increases and x_2 decreases. Note that it is not enough to focus only on x_1 because staying on an indifference curve (monotonic preferences or equivalently look at the utility function) means that a higher x_1 leads to a lower x_2. Consequently, the preferences are concave.

b) If the household only consumes good 1, he can afford and will buy $\frac{m}{p_1}$ units of good 1. Equivalently, if he only consumes good 2, he can afford and will buy $\frac{m}{p_2}$ units of good 2. Comparing the utility levels, the household only consumes good 1 if:

$$U(\frac{m}{p_1}, 0) = \left(\frac{m}{p_1}\right)^2 > 4\left(\frac{m}{p_2}\right)^2 = U\left(0, \frac{m}{p_2}\right)$$

or

$$p_2^2 > 4p_1^2$$

which is equivalent to our assumption $2p_1 < p_2$. Therefore, the indirect utility function is given by:

$$V(p, m) = \left(\frac{m}{p_1}\right)^2.$$

c) We have:

$$\overline{U} = \chi_1^2$$

and therefore:

$$\chi_1\left(p, \overline{U}\right) = \sqrt{\overline{U}}.$$

Alternative: Solving $V(p, m) = \left(\frac{m}{p_1}\right)^2$ for m, the expenditure necessary to achieve the utility $\overline{U}$ is given by:

$$e(p, \overline{U}) = p_1\sqrt{\overline{U}}.$$

Now we can either apply Shephard's lemma and differentiate the expenditure function with respect to p_1:

$$\frac{\partial(p, \overline{U})}{\partial p_1} = \chi_1(p, \overline{U}) = \sqrt{\overline{U}}$$

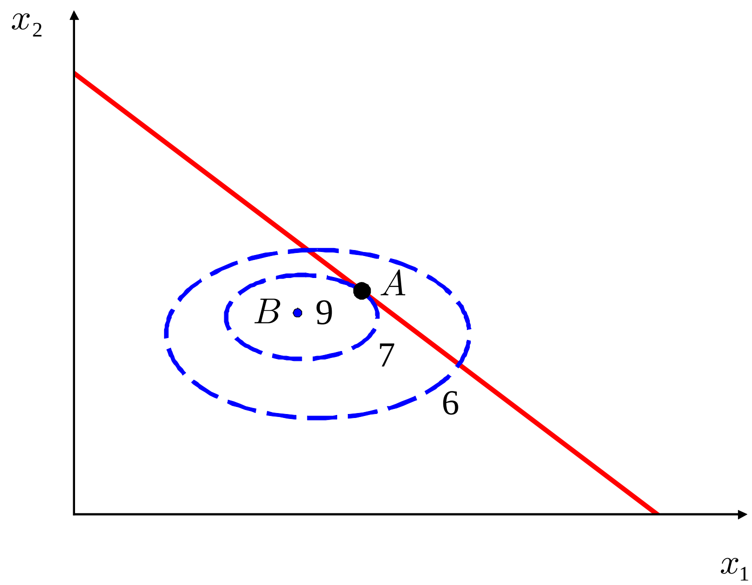
or replace m in the Marshallian demand by the expenditure e. The Marshallian demand is equal to:

$$x_1\left(m, p\right) = \frac{m}{p_1}.$$

After replacing m by $e(p, \overline{U})$, we receive:

$$\chi_1\left(p, \overline{U}\right) = \frac{p_1\sqrt{\overline{U}}}{p_1} = \sqrt{\overline{U}}.$$

Problem 7 (4 Points)



In the figure you see preferences with bliss point at B. One of these duality equations

$$\begin{aligned} e\left(p, V\left(p, m\right)\right) &= m, \\ V\left(p, e\left(p, \overline{U}\right)\right) &= \overline{U} \end{aligned}$$

is not valid. Which one? Why?

Solution

We assume that the straight line through A is the budget line for the budget m and the price vector p. Given the budget m and price vector p, the maximal utility is given by $V(p, m) = 9$ and achieved in B. For the given price vector p, the expenditure $e(p, V(p, m))$ needed to achieve the utility $V(p, m) = 9$ is lower than m because the household optimum B is below the budget line. Hence we know that $e(p, V(p, m)) < m$, i.e., the first duality equation does not hold.