Advanced Microeconomics

Hidden action

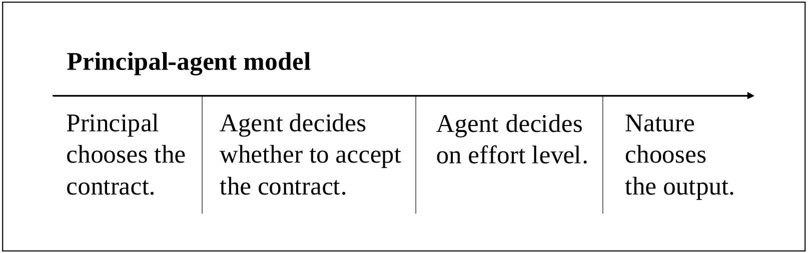
Harald Wiese

University of Leipzig

Hidden action

Introduction

- The agent is to perform some task for the principal, the asymmetry of information occurs after the agent has been employed
- Problem: the output is assumed to be a function of both the agent's effort and chance
- Since the effort is not observable, the payment to the agent (as specified in the contract) is a function of the output, but not of effort



Introduction

- The principal-agent problem is described as the principal's maximization problem subject to two conditions:
    - participation constraint
    - incentive compatibility
- Principal-agent models often assume that the principal is risk neutral and the agent risk averse;
- Pareto optimality requires that the agent does not bear any risk.
- However, in order to incite the agent not to be lazy, it may be necessary to have the agent bear some risk

The principal-agent model

Definition (Principal-agent problem)

A tuple $\Gamma = (\{P, A\}, \; E, X, (\xi_e)_{e \in E}, c, \overline{u})$ is called a principal-agent problem where

- P is the principal; A is the agent,
- $E = \mathbb{R}_+$ is the agent's action set (his effort level),
- $c : E \to \mathbb{R}$ is the agent's cost-of-effort function,
- X is the output set or the set of net profits,
- ξ_e is the probability distribution on X generated by effort level e,
- the principal's nonprobabilistic payoff is given by
$$x - w, \quad \text{with } x \in X, \text{ wage rate } w \in \mathbb{R},$$
- the agent's nonprobabilistic payoff is given by
$$w - c(e)$$
- the agent's reservation utility is $\overline{u} \in \mathbb{R}$.

Sequence, strategies, and solution strategy

The principal-agent problem is modeled as a four-stage game

1. The principal chooses a wage function which specifies the wage as a function of the output. This wage function is also called a contract
2. The agent decides whether to accept the contract
3. The agent decides on his effort level
4. Nature chooses the output and thus the payoffs for both principal and agent

Definition (Strategies)

Let Γ be a principal-agent problem. The principal's strategy is a wage function $s_P = w : X \to \mathbb{R}$. The agent's strategy is a function $s_A : S_P \to \{\mathbf{y}, \mathbf{n}\} \times E$, where **y** means ("yes" or "accept") and **n** ("no" or "decline") and refers to the agent's participation decision. s_A is sometimes written as $\left(s_A^{\{\mathbf{y},\,\mathbf{n}\}}, s_A^E\right)$ with $s_A^{\{\mathbf{y},\,\mathbf{n}\}}(s_P) \in \{\mathbf{y}, \mathbf{n}\}$ and $s_A^E(s_P) \in E$.

Sequence, strategies, and solution strategy

- The principal can foresee the agent's reaction to any wage function he offers
- We look for a subgame-perfect equilibrium
- Our solution strategy to the principal-agent problem focuses on the effort level of an agent who accepts a contract
- Imagine that the principal aims for an effort level $b \in E$, the principal maximizes his payoff under two conditions:
  - The agent needs to prefer accepting the contract and exerting effort level b to not accepting the contract
  - The agent needs to prefer effort level b to any other effort level $e \in E$

Observable effort

- The principal can directly observe the agent's effort or the principal observes the output and can deduce the effort unequivocally
- The principal can propose a payment scheme with domain E or X (we assume domain X)
- Assume that the principal wants the agent to choose some effort level $b \in E$; his maximization problem is

$$\max_{w} \left(x\left(b\right) - w\left(x\left(b\right)\right)\right)$$

subject to the side conditions

$$\begin{array}{lr} w\left(x\left(b\right)\right) - c\left(b\right) \geq \overline{u}, & \text{participation c.} \\ w\left(x\left(b\right)\right) - c\left(b\right) \geq w\left(x\left(e\right)\right) - c\left(e\right) \text{ for all } e \in E, & \text{incentive c.} \end{array}$$

- There is no need to give more to the agent than the reservation utility;

$$w\left(x\left(b\right)\right) = \overline{u} + c\left(b\right) \tag{1}$$

is the minimal wage that fulfills the participation constraint

Observable effort

- Thus, the optimal effort chosen by the principal (!) is

$$e^* = \arg\max_e \left(x(e) - (\overline{u} + c(e)) \right)$$

where e^* is obtainable (in good-natured problems) by

$$\underbrace{\frac{dx}{de}}_{\text{marginal output}} \overset{!}{=} \underbrace{\frac{dc}{de}}_{\text{marginal cost}} .$$

- Incentive constraint fulfilled by a boiling-in-oil contract:

$$w(x) = \begin{cases} \overline{u} + c(e), & x = x(e) \\ -\infty & x \neq x(e) \end{cases}$$

- The payoffs are $x(e^*) - \overline{u} - c(e^*)$ for the principal and $\overline{u}$ for the agent
- The sum of the payoffs is $x(e^*) - c(e^*)$ and hence the payoff that the principal could achieve if he were his own agent

Unobservable effort

The model

- We assume that the principal knows the probability distribution ξ_e generated by any effort level $e \in E$
- In general, this knowledge plus the specific output is not sufficient to reconstruct the effort level itself
- Principal bases his wage payments w on the output

Definition (Principal-agent model)

Let $\Gamma = \left(\{P, A\}, E, X, \ (\xi_e)_{e\in E}, c, u, \overline{u}\right)$ be a principal-agent problem. The principal-agent model with n outputs is given by

- the output set $X = \{x_1, ..., x_n\}$,
- the principal's utiliy function $u_P(s_P, s_A) =$
$$\begin{cases} \sum_{x\in X} \xi_{s_A^E(s_P)}(x)\,(x - w(x)), & s_A^{\{\mathbf{y},\,\mathbf{n}\}}(s_P) = \mathbf{y} \\ 0, & \text{otherwise} \end{cases}$$
- the agent's utility function $u_A(s_P, s_A) =$
$$\begin{cases} \sum_{x\in X} \xi_{s_A^E(s_P)}(x)\, u(w(x)) - c\left(s_A^E(s_P)\right), & s_A^{\{\mathbf{y},\,\mathbf{n}\}}(s_P) = \mathbf{y} \\ \overline{u}, & \text{otherwise} \end{cases}$$

  where $u : \mathbb{R} \to \mathbb{R}$ (not u_A) is a vNM utility function obeying $u' > 0$ and $u'' < 0$.

- The agent's utility function u_A is somewhat special; the cost of effort can be separated from the utility with respect to the wage earnings
- We now try to solve the principal-agent model. The two side conditions for action $b \in E$ are

$$\sum_{x \in X} \xi_b(x) u(w(x)) - c(b) \geq \overline{u}, \qquad \text{participation c.}$$

$$\begin{aligned} &\sum_{x \in X} \xi_b(x) u(w(x)) - c(b) \\ &\geq \sum_{x \in X} \xi_e(x) u(w(x)) - c(e) \text{ for all } e \in E, \end{aligned} \qquad \text{incentive c.}$$

Unobservable effort

Applying the Lagrangean method to the participation constraint

- First, we assume that the incentive constraint poses no problem
- Let $w_i := w(x_i)$ for all $i = 1, ..., n$; the principal's maximization problem is

$$\max_{w_1,...,w_n} \sum_{i=1}^{n} \xi_b(x_i)(x_i - w_i)$$

subject to the participation constraint

$$\sum_{i=1}^{n} \xi_b(x_i)\, u(w_i) - c(b) \geq \overline{u}.$$

- The principal maximizes his payoff by fulfilling the participation constraint as an equality

Unobservable effort

Applying the Lagrangean method to the participation constraint

- The Lagrangean of this problem is

$$\begin{aligned} & L\left(w_1, w_2, ..., w_n, \lambda\right) \\ = \; & \sum_{i=1}^{n} \xi_b\left(x_i\right)\left(x_i - w_i\right) + \lambda\left(\sum_{i=1}^{n} \xi_b\left(x_i\right) u\left(w_i\right) - c\left(b\right) - \overline{u}\right). \end{aligned}$$

- The Lagrange multiplier $\lambda > 0$ indicates the additional payoff accruing to the principal if the participation constraint is relaxed. Reducing the reservation utility by one unit increases the principal's payoff by

$$\lambda = -\frac{du_P}{d\overline{u}}$$

which is not quite, but basically correct

Unobservable effort

Applying the Lagrangean method to the participation constraint

- The partial derivatives with respect to w_i $(i = 1, ..., n)$ yield

$$\frac{\partial L}{\partial w_i} = \underbrace{-\xi_b(x_i)}_{\substack{\text{wage payments increase} \\ \text{with probability } \xi_b(x_i)}} + \lambda \underbrace{\xi_b(x_i)\, u'(w_i)}_{\substack{\text{participation constraint} \\ \text{is relaxed}}} \overset{!}{=} 0.$$

- Bad news: An increase of w_i (i.e., in case of output x_i) by one unit reduces the expected profit by $\xi_b(x_i)$ because the wage payments are increased by one unit with probability $\xi_b(x_i)$
- Good news: A wage increase eases the participation constraint by $\xi_b(x_i)\, u'(w_i)$; multiply by λ to obtain the profit increase
- The wages are the same for all outputs:

$$u'(w_i) \overset{!}{=} \frac{1}{\lambda}$$

the risk averse agent is not exposed to any risk

Unobservable effort

Applying the Kuhn-Tucker method to the incentive constraint

- A constant wage is not optimal if the incentive constraint is binding
- The principal's optimization problem leads to the Lagrangean

$$
\begin{aligned}
& L\left(w_1, w_2, ..., w_n, \lambda, \mu\right) \\
& = \sum_{i=1}^{n} \xi_b\left(x_i\right)\left(x_i - w_i\right) \\
& + \lambda\left(\sum_{i=1}^{n} \xi_b\left(x_i\right) u\left(w_i\right) - c\left(b\right) - \overline{u}\right) \text{ (participation constraint)} \\
& + \mu_{e'}\left(\sum_{x\in X} \xi_b\left(x\right) u\left(w\left(x\right)\right) - c\left(b\right) - \left(\sum_{x\in X} \xi_{e'}\left(x\right) u\left(w\left(x\right)\right) - c\left(e'\right)\right)\right) \\
& + \mu_{e''}\left(\sum_{x\in X} \xi_b\left(x\right) u\left(w\left(x\right)\right) - c\left(b\right) - \left(\sum_{x\in X} \xi_{e''}\left(x\right) u\left(w\left(x\right)\right) - c\left(e''\right)\right)\right) \\
& + \text{... (all the other incentive constraints)}
\end{aligned}
$$

Unobservable effort

Applying the Kuhn-Tucker method to the incentive constraint

- The Lagrange multipliers $\mu_{e'} > 0$, $\mu_{e''} > 0$ reflect the principal's marginal payoff for relaxing the incentive constraint with respect to effort e', e'' ...
- We cannot, in general, be sure that all the incentive c. are binding
- Kuhn-Tucker optimization theory says that the product

$$\mu_e \left(\sum_{x \in X} \xi_b(x)\, u(w(x)) - c(b) - \left(\sum_{x \in X} \xi_e(x)\, u(w(x)) - c(e) \right) \right)$$

has to be equal to zero for every effort level $e \in E$

- We differentiate the Lagrange function with respect to x_i to obtain

$$\frac{\partial L}{\partial w_i} = \underbrace{-\xi_b(x_i)}_{\substack{\text{wage payments increase} \\ \text{with probability } \xi_b(x_i)}} + \lambda \underbrace{\xi_b(x_i)\, u'(w_i)}_{\substack{\text{participation constraint} \\ \text{is relaxed}}}$$

$$+\mu_{e'} \underbrace{\overbrace{\left(\xi_b(x_i) - \xi_{e'}(x_i)\right)}^{\text{assumption: positive}} u'(w_i)}_{\substack{\text{incentive constraint} \\ \text{is relaxed}}} + \mu_{e''} \underbrace{\overbrace{\left(\xi_b(x_i) - \xi_{e''}(x_i)\right)}^{\text{assumption: negative}} u'(w_i)}_{\substack{\text{incentive constraint} \\ \text{is exacerbated}}} + \ldots \overset{!}{=} 0$$

Unobservable effort

Applying the Kuhn-Tucker method to the incentive constraint

- Assume the special case of two effort levels b and e
- The above maximization condition implies (after some reshuffling)

$$u'(w_i) \overset{!}{=} \frac{\xi_b(x_i)}{\lambda \xi_b(x_i) + \mu_e(\xi_b(x_i) - \xi_e(x_i))} = \frac{1}{\lambda + \mu_e \frac{\xi_b(x_i) - \xi_e(x_i)}{\xi_b(x_i)}}.$$

- Assume $\mu_e > 0$ and $\xi_b(x_i) > \xi_e(x_i)$. Then
  - wage w_i should be relatively high in order to give the agent an incentive to choose b rather than e
  - formally, $u'(w_i)$ is smaller for $\mu_e > 0$ than for $\mu_e = 0$
  - Sketch a concave vNM utility function so that you see why a small u' implies a large w_i.

Special case: two outputs

The model

- Two output levels, x_1 and x_2, and two actions, e and b
- We assume
  - Output x_2 is higher than output $x_1 : x_1 < x_2$,
  - b makes x_2 more likely than $e : \xi_b(x_2) > \xi_e(x_2)$,
  - b is the principal's preferred action

Exercise

Do $x_1 < x_2$ and $\xi_b(x_2) > \xi_e(x_2)$ imply that the principal aims for b rather than e?

Special case: two outputs

The model

- So far:
  - principal fixes wages $w = w(x)$ and
  - vNM utility $u(w)$
- From now on:
  - principal fixes vNM utility levels and
  - $w(u)$ is the wage level necessary in order to give vNM utility u to the agent
- If u is concave, $w = u^{-1}$ is convex.

The principal who aims at effort level b obtains maximal payoff

$$\pi(b) = \max_{u_1, u_2} \xi_b(x_1)[x_1 - w(u_1)] + \xi_b(x_2)[x_2 - w(u_2)]$$

subject to the two side conditions

$$\begin{array}{ll} \xi_b(x_1) u_1 + \xi_b(x_2) u_2 - c(b) \geq \overline{u}, & \text{p. c.} \\ \xi_b(x_1) u_1 + \xi_b(x_2) u_2 - c(b) \geq \xi_e(x_1) u_1 + \xi_e(x_2) u_2 - c(e), & \text{i. c.} \end{array}$$

Solving for u_2 yields

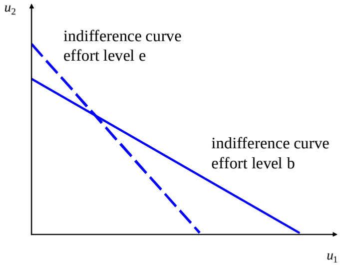
$$\begin{array}{ll} u_2 \geq \frac{\overline{u} + c(b)}{\xi_b(x_2)} - \frac{\xi_b(x_1)}{\xi_b(x_2)} u_1, & \text{participation c.} \\ u_2 \geq u_1 + \frac{c(b) - c(e)}{\xi_b(x_2) - \xi_e(x_2)}, & \text{incentive c.} \end{array}$$

Assuming a constant expected utility $\widetilde{u}$, the indifference curve for effort level e is given by

$$\widetilde{u} = \xi_e(x_1)\, u_1 + \xi_e(x_2)\, u_2 - c(e) \quad \text{or}$$

$$u_2 = \frac{\widetilde{u} + c(e)}{\xi_e(x_2)} - \frac{\xi_e(x_1)}{\xi_e(x_2)} u_1 .$$

By $\xi_b(x_2) > \xi_e(x_2)$ the indifference curves for b are flatter than those for e.

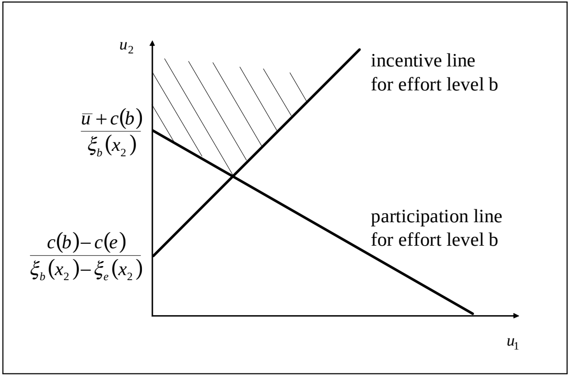


Interpretation of $\frac{\xi_e(x_1)}{\xi_e(x_2)}$?

Participation constraint for effort level b?

Special case: two outputs

The indifference curves



- $c(b) - c(e) > 0$ —> incentive line above 45°-line
- utiliy difference $u_2 - u_1$ does not fall below $\frac{c(b)-c(e)}{\xi_b(x_2)-\xi_e(x_2)}$
- utility levels u_1 and u_2 have to be chosen inside the highlighted area

Special case: two outputs

The principal's iso-profit lines

- The principal's profit

$$\pi(u_1, u_2) = \xi_b(x_1)[x_1 - w(u_1)] + \xi_b(x_2)[x_2 - w(u_2)],$$

- The slope of the iso-profit lines is given by

$$\frac{du_2}{du_1} = -\frac{\frac{\partial \pi}{\partial u_1}}{\frac{\partial \pi}{\partial u_2}} = -\frac{\xi_b(x_1)\, w'(u_1)}{\xi_b(x_2)\, w'(u_2)}$$

  - negatively sloped because $w'(u_1)$ and $w'(u_2)$ are positive
  - the nearer the iso-profit lines are to the origin, the higher the profit they indicate

Special case: two outputs

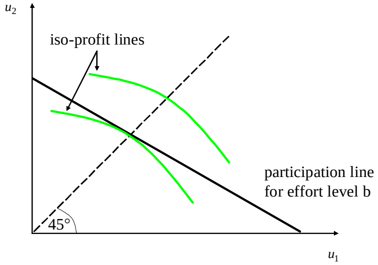
The principal's iso-profit lines

An increase in u_1 leads to

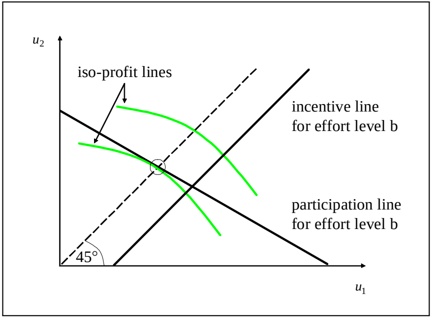
- an increase in $w'(u_1)$ (convexity of w),
- a decrease in u_2 (negative slope of the iso-profitline) and hence
- a decrease in $w'(u_2)$ (convexity of w)

—> absolute value of the slope increases

$u_1 = u_2$ —> iso-profit line's slope: $-\frac{\xi_b(x_1)}{\xi_b(x_2)}$



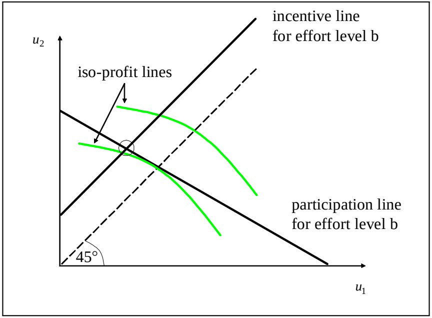
If we do not need to worry about incentive compatibility, ...



- $c_b \leq c_e \longrightarrow u_1 + \frac{c(b)-c(e)}{\xi_b(x_2)-\xi_e(x_2)} \leq u_1$
- The incentive constraint does not prevent the first-best solution (i.e., the solution when there is no asymmetric information)

Special case: two outputs

Solving the principal-agent problem



- $c_b > c_e \longrightarrow u_1 + \frac{c(b)-c(e)}{\xi_b(x_2)-\xi_e(x_2)} > u_1$
- Optimal risk sharing at $u_1 = u_2$ is not possible
- Second-best solution (taking asymmetric information into account)

From Milgrom/Roberts (1992, pp. 200-203):

- We have two outputs 10 and 30.
- The agent has two effort levels, 1 and 2. Effort level 2 makes output 30 more likely than effort level 1 :

Effort level	Output $x = 10$	Output $x = 30$
$e = 1$	$\xi_1(10) = 2/3$	$\xi_1(30) = 1/3$
$e = 2$	$\xi_2(10) = 1/3$	$\xi_2(30) = 2/3$

- The agent is risk averse with vNM utility function $u(w, e) = \sqrt{w} - (e - 1)$. The reservation utility is $\overline{u} = 1$.
- The principal has the profit function π given by $\pi(w, x) = x - w$.
- In case of unobservable effort, the principal's wage function is given by $w(10) \equiv w_l$, $w(30) \equiv w_h$.

Special case: two outputs

Solving the principal-agent problem: observable effort (questions)

- If the principal aims for $e = 1$, what is his optimal wage function?
- If the principal aims for $e = 2$, what is his optimal wage function?
- Should the principal aim for effort level 1 or 2?

If the principal aims for $e = 1$, he needs to take care of the participation constraint, only:

$$\sqrt{w} - (e - 1) \geq \overline{u}.$$

The wage rate $w = 1$ fulfilling this constaint automatically takes care of the incentive problem.

Special case: two outputs

Solving the principal-agent problem: observable effort (answers)

In case of observable effort, it is easy to force $e = 2$. The wage rate of $w_{e=2} = 4$ guarantees the participation constraint $\sqrt{w_{e=2}} - (2-1) \geq 1$. The incentive constraint is $\sqrt{w_{e=2}} - (2-1) \geq \sqrt{w_{e=1}} - (1-1)$ which can be rewritten as

$$\begin{aligned} \sqrt{w_{e=1}} &\leq \sqrt{w_{e=2}} - 1 \\ &= \sqrt{4} - 1 \\ &= 1. \end{aligned}$$

Thus, the wage function

$$w = \begin{cases} 4, & e = 2 \\ 1, & e = 1 \end{cases}$$

is optimal.

$e = 1$ and $w = 1$ implies the expected profit

$$\begin{aligned}\pi(e=1) &= \frac{2}{3}\cdot 10 + \frac{1}{3}\cdot 30 - 1 \\ &= \frac{47}{3}\end{aligned}$$

while $e = 2$ and $w = 4$ leads to

$$\begin{aligned}\pi(e=2) &= \frac{1}{3}\cdot 10 + \frac{2}{3}\cdot 30 - 4 \\ &= \frac{58}{3} \\ &> \frac{47}{3}.\end{aligned}$$

The principal should aim for $e = 2$.

Special case: two outputs

Solving the principal-agent problem: unobservable effort for e=2 (questions)

- Write down the participation constraint in terms of $\sqrt{w_l}$ and $\sqrt{w_h}$.
- Write down the incentive constraint in terms of $\sqrt{w_l}$ and $\sqrt{w_h}$.
- Depict the two constraints by putting $\sqrt{w_l}$ on the abscissa and $\sqrt{w_h}$ on the ordinate.
- Determine w_l and w_h!

Special case: two outputs

Solving the principal-agent problem: unobservable effort for e=2 (answers)

In case of unobservability, the wage needs to be a function of output, not effort. w_l is the wage for the low output 10 and w_h is the wage for the high output 30.

The agent's participation constraint for the high effort 2 is

$$\begin{aligned}
& \frac{1}{3}u(w_l,2)+\frac{2}{3}u(w_h,2) \\
= & \frac{1}{3}\left(\sqrt{w_l}-1\right)+\frac{2}{3}\left(\sqrt{w_h}-1\right) \\
= & \frac{1}{3}\sqrt{w_l}+\frac{2}{3}\sqrt{w_h}-1 \\
\geq & 1,
\end{aligned}$$

or

$$\sqrt{w_h}\geq 3-\frac{1}{2}\sqrt{w_l}.$$

The incentive constraint for effort 2 rather than 1 is

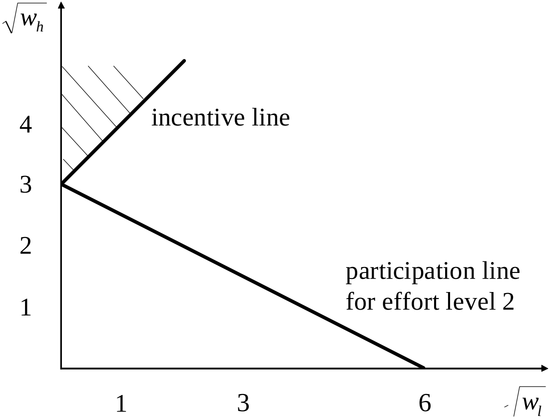
$$\begin{aligned} & \frac{1}{3}\sqrt{w_l} + \frac{2}{3}\sqrt{w_h} - 1 \\ = & \ \frac{1}{3}u\left(w_l, 2\right) + \frac{2}{3}u\left(w_h, 2\right) \\ \geq & \ \frac{2}{3}u\left(w_l, 1\right) + \frac{1}{3}u\left(w_h, 1\right) \\ = & \ \frac{2}{3}\sqrt{w_l} + \frac{1}{3}\sqrt{w_h}, \end{aligned}$$

which can also be written as

$$\sqrt{w_h} \geq 3 + \sqrt{w_l}.$$

constraints square

root



7.pdf

From the figure, we learn that the principal should not pay a positive wage to the agent in case of $x = 10$. We have $\sqrt{w_h} = 3$ and $\sqrt{w_l} = 0$ or the wage function

$$w = \begin{cases} 9, & x = 30 \\ 0, & x = 10 \end{cases}.$$

The principal's profit is

$$\begin{aligned} \pi(e=2) &= \frac{1}{3} \cdot (10-0) + \frac{2}{3} \cdot (30-9) \\ &= \frac{52}{3}. \end{aligned}$$

Is the principal's profit higher for $e = 1$ than for $e = 2$?
Very similar to the case of observable effort, if the effort level 1 is aimed for, the incentive constraint is no problem. We know that $w = 1$ fulfills the participation constraint and leads to the profit $\frac{47}{3}$. By $\frac{52}{3} > \frac{47}{3}$ the principal should go for $e = 2$. Note $\frac{58}{3} > \frac{52}{3}$, i.e., observability leads to a higher profit. After all, $e = 2$ is a second-best solution, only.

Special case: two outputs

Solving the principal-agent problem: unobservable effort (question different problem)

What is the optimal contract for these probabilities:

Effort level	Output $x = 10$	Output $x = 30$
$e = 1$	$\xi_1(10) = 2/3$	$\xi_1(30) = 1/3$
$e = 2$	$\xi_2(10) = 0$	$\xi_2(30) = 1$

The new probabilities reduce the principal's uncertainty. The high effort precludes the low output. Here, a boiling-in-oil contract is optimal:

$$w = \begin{cases} 4, & x = 30 \\ 0, & x = 10 \end{cases}$$

fulfills the participation constraint because the agent has the (expected) payoff $\sqrt{4} - (2-1) = 1 = \bar{u}$. Effort level $e = 1$ leads to the expected utility $\frac{2}{3}\sqrt{0} + \frac{1}{3}\sqrt{4} = \frac{2}{3} < 1$.

More complex principal-agent structures

- We consider two-tier principal-agent structures. *Tirole (1986)* points to three-tier structures

	principal	supervisor	agent
production unit	manager	foreman	worker
regulation	government	regulating authority	firm
PhD procedure	faculty council	professor	PhD stud.
professorship	ministry of educ.	dean/rector	professor

- time, competence or cost efficiency
- Does the supervisor act in the principal's interests? Sometimes,
  - the agent's achievements reflect on the supervisor,
  - the supervisor and the agent collude against the principal,
  - secret side payments play a role.