UNIVERSITY OF LEIPZIG
FACULTY OF ECONOMICS AND MANAGEMENT SCIENCE

MASTER EXAM

DATE:	23.02.2024
MODULE:	**Advanced Microeconomics**
DURATION:	120 minutes
EXAMINER:	**Prof. Dr. Harald Wiese**

ENROLMENT-No.:

MASTER DEGREE COURSE:

NAME, FIRST NAME:

SIGNATURE (STUDENT):

ANNOTATIONS:
Do not use your own paper!
Write your name and your enrollment number on every single sheet.
Give reasons for your answers! Underline your solutions!
Write legibly, please! Solve all the exercises!
Sie können auf Deutsch schreiben!/You can write in English!

ALLOWED TOOLS: None

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	$\sum$	**Grade**
Pkt:																

DATE, SIGNATURE (EXAMINER):

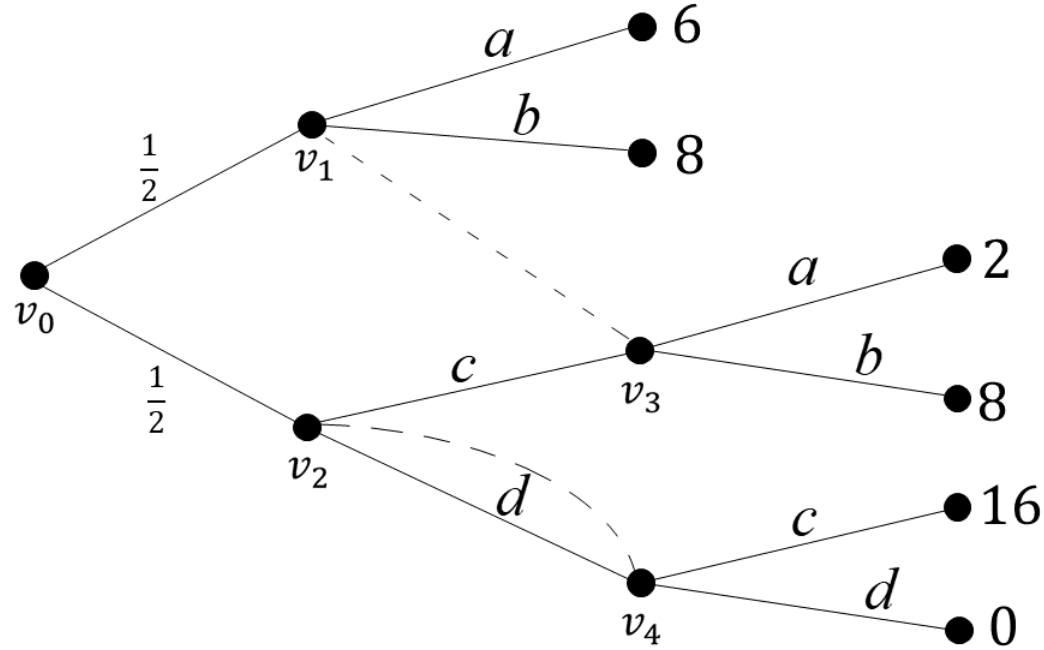
Problem 1 (6 points)
Let (N, v) be a cooperative game with player set $N = \{1, 2, 3\}$ and the coalition function

$$v(K) = \begin{cases} 2, & |K| = 2 \\ 0, & \text{otherwise} \end{cases}$$

Examine whether the Shapley payoff vector lies in the core.

Solution
Since the coalition function depends on the number of players only, all players are symmetric. By symmetry and efficiency, $Sh_1 = Sh_2 = Sh_3 = 0$. The payoff vector $(0, 0, 0)$ does not lie in the core because it is blockable by any two-player coalition. For example, we have $v(\{1, 2\}) = 2 > 0 = Sh_1 + Sh_2$.

Problem 2 (11 points)
Consider the following decision problem:



a) State all pure strategies.

b) Determine the best behavioral strategies.

Solution

a) There are $2^2 = 4$ strategies: $\lfloor a, c \rfloor$, $\lfloor a, d \rfloor$, $\lfloor b, c \rfloor$, and $\lfloor b, d \rfloor$.

b) Let α, β denote the probability weights put on action a and c, respectively. A behavioral strategy is defined by $s = (\alpha, \beta)$. At information set $I(\{v_1, v_3\})$, it is reasonable to choose action b to reach 8. That implies that $\alpha = 0$. At $\{v_2, v_4\}$, the payoff of 16 is reached with positive probability if and only if $0 < \beta < 1$. In this case, we find the best behavioral strategy by maximizing the expected payoff

$$\begin{aligned} u(0, \beta) &= \frac{1}{2}8 + \frac{1}{2}[\beta 8 + (1-\beta)\beta 16] \\ &= 4 + \frac{1}{2}\left[24\beta - 16\beta^2\right] \\ &= 4 + \left[12\beta - 8\beta^2\right] \end{aligned}$$

over β. Maximization leads to

$$\begin{aligned}
\frac{\partial u(0,\beta)}{\partial \beta} &= 12 - 16\beta \stackrel{!}{=} 0 \\
\Rightarrow \beta^* &= \frac{3}{4}.
\end{aligned}$$

The second derivative ($\frac{\partial^2 u(0,\beta)}{\partial \beta^2} = -16 < 0$) confirms that $\beta^* = \frac{3}{4}$ indeed identifies a maximum. So, $s^* = (0, \frac{3}{4})$ is the best behavioral strategy. [The expected payoff of $s^* = (0, \beta^*)$ is found to be $u(s^*) = 4 + 12\frac{3}{4} - 8\frac{9}{16} = 8.5$. For $\beta = 0$, we reach a payoff of $4 < 8,5$; for $\beta = 1$, the payoff $8 < 8.5$. This is also the highest payoff that can be reached in pure strategies.]

Problem 3 (6 points)
Consider the preferences represented by the utility function

$$U(x_1, x_2) = x_1 - \sqrt{x_2}.$$

Comment on local non-satiation and convexity!

Solution
A preference relation $\succsim$ obeys local non-satiation if, for all bundles y, there exists a bundle x satisfying $x \succ y$ within an arbitrary ε-ball with center y. Since $MU_1 = \frac{\partial U(x_1,x_2)}{\partial x_1} = 1$, the utility function $U(x_1, x_2)$ represents preferences that obey local non-satiation. Additionally, we have $MU_2 = \frac{\partial U(x_1,x_2)}{\partial x_2} = -\frac{1}{2\sqrt{x_2}}$. The preferences are, thus, monotonic in x_1 but not monotonic in x_2 resulting in a positive slope of the indifference curves. The better set is below and right of an indifference curve. The slope of an indifference curve is $MRS = 2\sqrt{x_2}$. Since x_2 increases along the indifference curve as x_1 increases, the better set is not convex and, thus, the preferences are also not convex.

Problem 4 (10 points)
Consider the utility function $U(x_1, x_2) = (x_1 + x_2)^3$. Assume that both goods are equally expensive.

a) Derive the indirect utility function.

b) Derive the expenditure function.

c) Determine the compensating variation for a price increase to $p_1 = p_2 = 4$ if the current prices are $p_1 = p_2 = 2$ and money budget is $m = 12$.

Solution

a) A monotonic transformation yields $W(x_1, x_2) = x_1 + x_2$. The goods are perfect substitutes. By $MRS = 1 = MOC$, we know that each bundle on the budget line is optimal, i.e. $x_1 + x_2 = \frac{m}{p}$ with $p = p_1 = p_2$. Indirect utility is, thus, $V(p, m) = \left(\frac{m}{p}\right)^3$.

b) Since U satisfies local non-satiation, we can use duality. Then, we have

$$\begin{aligned} \bar{U} &= V(p, e(p, \bar{U})) \\ \bar{U} &= \left(\frac{e(p, \bar{U})}{p}\right)^3 \\ \bar{U}^{\frac{1}{3}} &= \frac{e(p, \bar{U})}{p} \\ \Rightarrow \qquad e(p, \bar{U}) &= p\bar{U}^{\frac{1}{3}} \end{aligned}$$

Alternatively, we could derive Hicksian demand from $U(x_1, x_2) = (x_1 + x_2)^3$: $\chi_1(p, \bar{U}) + \chi_2(p, \bar{U}) = \bar{U}^{\frac{1}{3}}$. The expenditure function is given by

$$e(p, \bar{U}) = p\chi_1(p, \bar{U}) + p\chi_2(p, \bar{U}) = p\left(\chi_1(p, \bar{U}) + \chi_2(p, \bar{U})\right) = p\bar{U}^{\frac{1}{3}}.$$

c) The compensating variation is a monetary transfer to compensate for the event of the price doubling. To ensure the old utility level of $V(p^{old} = 2, m = 12) = 6^3$, the money budget should be doubled if prices double such that $V(p^{new} = 4, m + cv) = \left(\frac{12+cv}{4}\right)^3 = 6^3$. The compensating variation is $CV = |cv| = 12$.

Problem 5 (8 points)
Consider an exchange economy with two agents, A and B, and only one good. Both agents have strictly monotonic preferences. The initial endowment is given by $\omega = (35, 65)$.
State, algebraically,

a) the contract curve,

b) ω's exchange lens,

c) the core, and

d) an envy-free allocation.

Solution

a) In an exchange economy where both agents have strictly monotonic preferences, each feasible ($x^A + x^B \leq 100$) and efficient ($x^A + x^B \geq 100$) allocation is Pareto-efficient, because no agent can improve without making the other agent worse off. The contract curve is the locus of all Pareto-efficient allocations (the whole Edgeworth-line). Hence, the contract curve is given by $\left\{(x^A, 100 - x^A) : x^A \in [0, 100]\right\}$.

b) The exchange lens of ω consists of the initial endowment only.

c) The core (intersection of contract curve and exchange lens) is thus given by the initial endowment ω.

d) An allocation is envy-free if nobody strictly prefers the bundle of any other person. If both agents have strictly monotonic preferences, they do not envy the other agent if they receive the same bundle: $x^A = x^B = \frac{\omega^A + \omega^B}{2} = 50$. $(50, 50)$ is, thus, an envy-free allocation.

Problem 6 (4 points)
Consider the following two-person game with mixed strategies. Determine both reaction functions and all equilibria in mixed strategies.

		player 2	
		l	r
player 1	o	$(1,4)$	$(1,4)$
	u	$(1,4)$	$(1,4)$

Solution
Let σ_1 denote the probability of player 1 to play o and let σ_2 denote the probability of player 2 to play l. Player 1 is indifferent between o and u, independently of player 2's strategy. Similarly, Player 2 is indifferent between l and r, independently of player 1's strategy.We get

$$\sigma_1^R(\sigma_2) = [0,1], \quad \forall \sigma_2$$

and

$$\sigma_2^R(\sigma_1) = [0,1], \quad \forall \sigma_1$$

Hence, there are infinitely many equilibria in mixed strategies. The set of all equilibria is given by $\{(\sigma_1, \sigma_2) : \sigma_1 \in [0,1] \wedge \sigma_2 \in [0,1]\}$.
Four of them are in pure strategies $(\sigma_1, \sigma_2) = (0,0), (0,1), (1,0), (1,1)$.

Problem 7 (8 points)
Consider the following production function: $y = f(x_1, x_2) = \sqrt{x_1} x_2$.

a) Explore whether the production function exhibits decreasing, constant, or increasing returns to scale.

b) Is divine production possible?

c) In the short run factor 2 is fixed at level $\bar{x}_2 > 0$. Determine the short-run marginal cost function.

Solution:

a) The production function exhibits increasing returns to scale:

$$\begin{aligned} f(tx_1, tx_2) &= \sqrt{(tx_1)} \cdot (tx_2) \\ &= t^{\frac{1}{2}} \sqrt{x_1} \cdot tx_2 \\ &= t^{\frac{3}{2}} f(x_1, x_2) > tf(x_1, x_2) \end{aligned}$$

for every $t > 1$.

b) Divine production is not possible because $f(0,0) = 0$.

c)

$$C_s(y, \bar{x}_2) := \min_{x_1} \{w_1 x_1 + w_2 \bar{x}_2 : \sqrt{x_1} \bar{x}_2 \geq y\}$$

The firm uses $\bar{x}_2$. Nonetheless, the firm will use a positive amount of factor x_1 to realize a positive output ($y > 0$): $x_1^* = \frac{y^2}{\bar{x}_2^2}$.

$$\begin{aligned} C_s(y, \bar{x}_2) &= w_1 x_1(y) + w_2 \bar{x}_2 = \frac{w_1 y^2}{\bar{x}_2^2} + w_2 \bar{x}_2 \\ MC_s(y, \bar{x}_2) &= \frac{2w_1}{\bar{x}_2^2} y \end{aligned}$$

Problem 8 (6 points)
Let w and ω be an arbitrary state of the world and an arbitrary belief, respectively. Show or disprove that

a) if $s \in s^{R,\Omega}(\omega)$, then s is a weakly dominant strategy;

b) if s is a weakly dominant strategy, then $s \in s^{R,W}(w)$.

Solution

a) This assertion is false. Being a best response with respect to Ω means that there exists a belief $\omega \in \Omega$ such that (viewing s as a trivial mixed strategy σ) $\sigma \in \sigma^{R,\Omega}(\omega)$ is true. However, it does not mean that $\sigma \in \sigma^{R,\Omega}(\omega)$ is true for all beliefs, which is a necessary condition for weak dominance. An appropriate counter-example is given by:

	w_1	w_2
s_1	2	1
s_2	1	2

Here, s_2 is a best response with respect to Ω, e.g. for $\omega = \left(\frac{1}{4}, \frac{3}{4}\right)$, but it is not a weakly dominant strategy.

b) If s is a weakly dominant strategy, then $u(s, w) \geq u(s', w)$ holds for all $w \in W$ and every strategy $s' \neq s$. This implies that there exists a $w \in W$ such that $s \in s^{R,W}(w)$ is true, which means that s is rationalizable with respect to W. So, assertion **b)** is shown.

Problem 9 (6 points)
Consider the following two-person game that is repeated twice. The discount factor is $\delta = 1$ for both players. Strategies are written as quintuples $\lfloor a, a_{TT}, a_{TF}, a_{FT}, a_{FF} \rfloor$ where a is the action (T or F) at the first stage and a_{TF} the action at the second stage if she chose T at the first stage and he F.

		he	
		T	F
she	T	$(8, 6)$	$(3, 3)$
	F	$(5, 5)$	$(6, 8)$

a) Is $(\lfloor F, T, T, T, F \rfloor, \lfloor F, F, T, T, F \rfloor)$ a Nash equilibrium of the twice-repeated game?

b) Is $(\lfloor F, T, T, T, F \rfloor, \lfloor F, F, T, T, F \rfloor)$ a subgame-perfect Nash equilibrium of the twice-repeated game?

Solution:

a) If $s = (s_{she}, s_{he}) = (\lfloor F, T, T, T, F \rfloor, \lfloor F, F, T, T, F \rfloor)$ is played, he gets his maximum payoff $u_{he}(s) = 8 + 8 = 16$, while she gets $u_{she}(s) = 6 + 6 = 12$. If she already deviates in the first period by playing either $s'_{she} = \lfloor T, \cdot, T, \cdot, \cdot \rfloor$ or $s''_{she} = \lfloor T, \cdot, F, \cdot, \cdot \rfloor$, she gets $u_{she}((s'_{she}, s_{he})) \leq 3 + 8 = 11 < 12$ or $u_{she}((s''_{she}, s_{he})) \leq 3 + 5 = 8 < 12$. If she deviates in the second period only by playing $s_{she} = \lfloor F, \cdot, \cdot, \cdot, T \rfloor$, she gets $u_{she}((s'_{she}, s_{he})) = 6 + 3 = 9 < 12$. Hence, s a Nash equilibrium.

b) Consider the subgame in stage 2 after TT has been played in stage 1. If he deviates by playing $a_{TT} = T$, he gets 6, rather than 3, in this subgame. Hence, s is no subgame-perfect Nash equilibrium.

Problem 10 (8 points)
Consider the following model of a polypsonistic labor market with education where each worker (no unemployment) chooses his education $a \in \{0,1\}$ after the screening principal P (or principals) offers two wage rates, $w_0 \in \mathbb{R}$ for workers with education $a = 0$ and $w_1 \in \mathbb{R}$ for workers with education $a = 1$. The proportion of workers with productivity $t_h = 6$ is given by $\tau_h = \frac{1}{3}$, the proportion of workers with productivity $t_l = 3$ by $\tau_l = \frac{2}{3}$. The payoff for worker $t \in \{t_l, t_h\}$ with education $a \in \{0,1\}$ is given by

$$u_t(w_a, a) = w_a - \frac{12}{t} \cdot a,$$

while the principal's non-probabilistic payoff for employing worker t with education a is given by

$$u_P(t, w_a) = t - w_a.$$

Determine a pooling equilibrium!
Hint: The expected payoff of the principal is zero in equilibrium.

Solution:
We search for a pooling equilibrium where worker $t = t_l$ chooses $a = 0$ and worker $t = t_h$ chooses $a = 0$. Since the principal's expected payoff is zero (and no worker will have education $a = 1$ in the pooling equilibrium), the principal must offer

$$w_0 \stackrel{!}{=} E[t|a=0] = \frac{2}{3} \cdot 3 + \frac{1}{3} \cdot 6 = 4$$

to workers with education $a = 0$. Each worker is incited to choose $a = 0$, rather than $a = 1$, if

$$\begin{aligned} &(i) \quad u_{t_l}(w_0, 0) = 4 - 0 \geq w_1 - 4 = u_{t_l}(w_1, 1), \\ &(ii) \quad u_{t_h}(w_0, 0) = 4 - 0 \geq w_1 - 2 = u_{t_h}(w_1, 1) \end{aligned}$$

hold. This yields $w_1 \leq 6 = \min(8, 6)$. Hence, $(w_0, w_1) = (4, 5)$ and $a = 0$ for $t = t_l$ and $a = 0$ for $t = t_h$ is a pooling equilibrium. **Remark:** More precisely, the workers react on the two wage rates offered by the principle. Hence, a pooling equilibrium is given by the tuple $(\hat{w}, \hat{a})$ where $\hat{w} = (w_0, w_1) = (4, 5)$ and

$$\hat{a}(t, w_0', w_1') = \text{argmax}_{a \in \{0,1\}} u_t(w_a', a)$$

for all $t \in \{t_l, t_h\}$, $w_0' \in \mathbb{R}$, and $w_1' \in \mathbb{R}$.

Problem 11 (10 points)
Consider a first-price auction for an indivisible good with two risk-neutral bidders, 1 and 2. Bidder i's ($i \in \{1, 2\}$) willingness to pay for the good, t_i, is drawn independently from the uniform distribution on $[0, 1]$; t_i is known to bidder i but unknown to the other bidder. Show that the strategy combination (s_1, s_2) where $s_i : [0, 1] \to \mathbb{R}$, $t_i \mapsto s_i(t_i) = \frac{t_i}{2}$ ($i \in \{1, 2\}$) is an equilibrium.
Hint: You may want to follow these steps: Show that, for bidder 1 with type $t_1 \in [0, 1]$,
a) bidder 1's bid $b = 0$ is weakly better than all bids $b < 0$,
b) his bid $b = 1/2$ is weakly better than all bids $b > 1/2$,
c) for his remaining bids $b \in [0, 1/2]$, bidder 1 wins the auction with probability $2b$.

Solution:
Bidder 1 with type t_1 wins the auction, and thus the payoff $t_1 - b$, if his bid b is higher than his opponent's bid $t_2/2 \in [0, \frac{1}{2}]$, or equivalently, if $t_2 < 2b$. *i*) For $b \leq 0$, he never wins the auction. Hence, the bid $b = 0$ is weakly better than all bids $b < 0$. *ii*) For $b > 1/2$, he always wins the auction. Thus, by lowering his bid $b \in (1/2, \infty)$, he increases his expected payoff. Hence, $b = 1/2$ is weakly better than all bids $b > 1/2$. *iii*) For the remaining bids $b \in [0, 1/2]$, the bidder wins the auction with probability

$$p(b) = \int_0^{2b} dt_2 = 2b,$$

and his expected payoff is given by

$$(t_1 - b)2b.$$

By taking the first order derivative, we get the first-order condition

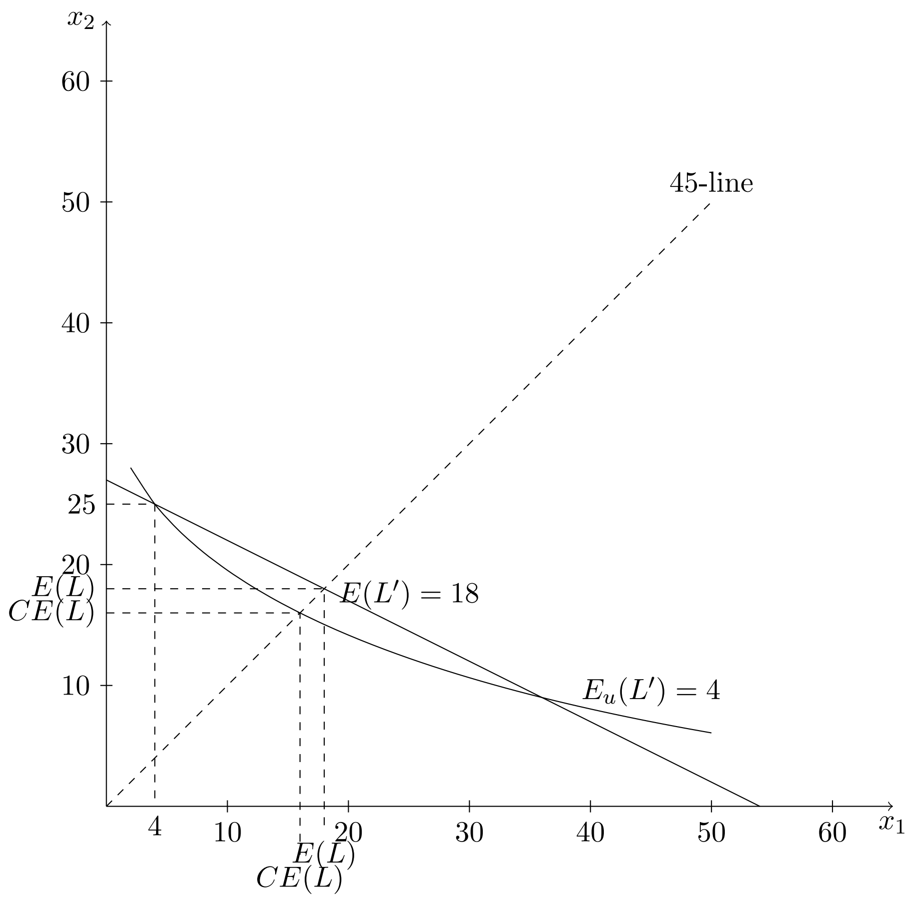
$$2t_1 - 4b \stackrel{!}{=} 0,$$

which is solved by $b = \frac{t_1}{2} \in [0, 1/2]$. Since $b = \frac{t_1}{2}$ maximizes the expected payoff of bidder 1 with type t_1, s_1 is bidder 1's best response to bidder 2's strategy s_2. Since the two players are symmetric, s_2 is a best response to s_1. Hence, (s_1, s_2) is an equilibrium.

Problem 12 (8 points)
Consider the vNM-utility function $u(x) = \sqrt{x}$ and the lottery $L = \lfloor 4; 25; \frac{1}{3}, \frac{2}{3} \rfloor$. Derive **graphically** in the graph below

- the expected value of the lottery $E(L)$,
- the certainty equivalent $CE(L)$, and
- the risk premium $RP(L)$.

Explain whether the utility function exhibits risk-averse, risk-neutral, or risk-loving preferences.



Solution:

See graph for $E(L)$ and $CE(L)$. The risk premium, $RP(L) = E(L) - CE(L)$, is not drawn in the graph. It corresponds to the difference between $E(L)$ and $CE(L)$ and could be indicated by a bracket/line/etc ranging from point $(16, 0)$ to $(18, 0)$. The agent is risk averse because $E(L) > CE(L)$ (or $u''(x) = -\frac{1}{4}x^{-3/2} < 0$).

Problem 13 (8 points)
Consider an exchange economy with two agents and three goods.

a) Assume that both agents have strictly monotonic preferences. Show that $p \cdot z(p) = 0$.

b) Assume $p \cdot z(p) = 0$ holds. Show that if all markets but one are cleared, the last one also clears or its price is zero (market clearance lemma).

Solution:

a) We show that

$$p \cdot z(p) = \sum_{i=1}^{2} p \cdot (x^i - w^i) = 0$$

holds, x^i being the household optimum and w^i being the initial endowment of agent i. *Proof:* x^i must be contained in the budget set. Hence, $p \cdot x^i > p \cdot w^i$ can be excluded. Assume $p \cdot x^i < p \cdot w^i$. The agent can afford bundles sufficiently close to x^i. Due to strict monotonicity, within the set of those affordable bundles, a bundle y^i exists that the household strictly prefers to x^i. This is a contradiction to x^i being a household optimum. Hence, $p \cdot x^i = p \cdot w^i$ which proves the claim.

b) Let n be the number of markets, p_j the price of good j, and $z_j(p)$ the excess demand on market j. Except the market for good k, $n-1$ markets are cleared. Thus $z_j(p)$=0, for all $j \neq k$. Hence,

$$0 = p \cdot z(p) = \sum_{j=1}^{n} p_j \cdot z_j(p) = \sum_{j \neq k} p_j \cdot z_j(p) + p_k \cdot z_k(p) = p_k \cdot z_k(p).$$

According to the above equation, either $p_k = 0$ or $z_k(p) = 0$ must hold.

Problem 14 (11 points)
Consider two firms, 1 and 2, that compete simultaneously in quantities. Firm 1's cost function is given by $C_1(x_1) = \frac{1}{2}x_1^2$, firm 2's by $C_2(x_2) = 30x_2$. Inverse demand is given by $p(x_1 + x_2) = 90 - x_1 - x_2$. The government imposes the unit tax t.

a) Determine the Cournot quantities depending on t.

b) Determine the unit tax that maximizes tax revenue by the government.

Solution:

a) The two profit functions are given by

$$\Pi_1(x_1, x_2) = (90 - x_1 - x_2)\, x_1 - \frac{1}{2}x_1^2 - tx_1,$$
$$\Pi_2(x_1, x_2) = (90 - x_1 - x_2)\, x_2 - 30x_2 - tx_2.$$

The two first-order conditions (FOCs) are given by

$$\frac{d\Pi_1(x_1, x_2)}{dx_1} = 90 - 2x_1 - x_2 - x_1 - t = 90 - 3x_1 - x_2 - t \overset{!}{=} 0,$$
$$\frac{d\Pi_2(x_1, x_2)}{dx_2} = 90 - x_1 - 2x_2 - 30 - t = 60 - x_1 - 2x_2 - t \overset{!}{=} 0.$$

By multiplying the first FOC by two and subtracting the second FOC, we get

$$120 - 5x_1 - t = 0$$
$$\Rightarrow x_1^C = 24 - \frac{t}{5}.$$

By substituting x_1^C into the first FOC, we get

$$x_2^C = 90 - 3\left(24 - \frac{t}{5}\right) - t = 18 - \frac{2}{5}t.$$

b) Total output is given by $X^C = x_1^C + x_2^C = 42 - \frac{3}{5}t$. Tax revenue is given by

$$X^C \cdot t = \left(42 - \frac{3}{5}t\right) t.$$

The first-order condition

$$\frac{d\left(X^C \cdot t\right)}{dt} = 42 - \frac{6}{5}t \overset{!}{=} 0$$

yields the tax-maximizing unit tax $t = 35$.