Advanced Microeconomics

Static Bayesian games

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Part E. Bayesian games and mechanism design

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2. The revelation principle and mechanism design

Static Bayesian games

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Conditional probability

Definition

Given:

- nonempty set M,
- probability distribution *prob* on M;
- events A and B .

If $prob\,(B) \neq 0$, the conditional probability of A given B is defined by

$$prob\,(A|B) = \frac{prob\,(A \cap B)}{prob\,(B)}.$$

If $prob\,(A|B) = prob\,(A)$, A and B are independent.

Problem

Throw a dice. What is the conditional probability of 1, 2 *or* 3 *pips (spots) if the number of pips is odd.*

Problem

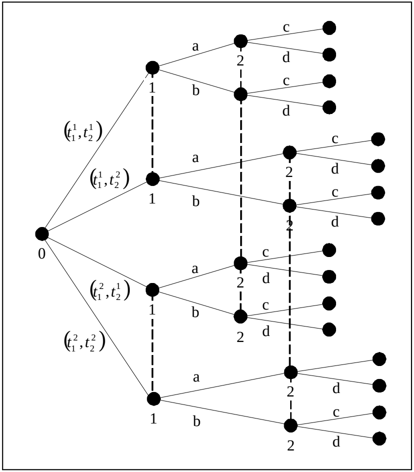
If events A and B are independent what about the probability of $A \cap B$?

- Simultaneous bids where
  - the highest bidder obtains the object and
  - pays his bid.
- Two bidders 1 and 2 who know their own willingness to pay (their "type").
- Sequence:
  - Nature decides the players' types.
  - Every player learns his own type and can condition his bid on his own type.
- Static Bayesian games
  = all players act simultaneously after learning their own types.

Definitions

Static Bayesian game – example for extensive form

- $N = \{1, 2\}$,
- $T_1 = \{t_1^1, t_1^2\}$ and $T_2 = \{t_2^1, t_2^2\}$,
- $A_1 = \{a, b\}$ and $A_2 = \{c, d\}$.
- At the initial node, nature chooses a type combination $t = (t_1, t_2) \in T_1 \times T_2$.
- $|T_1|$ informations sets for player 1



Definition (Static Bayesian game)

A static Bayesian game is a quintuple

$$\Gamma = \left(N, (A_i)_{i\in N}, (T_i)_{i\in N}, \tau, (u_i)_{i\in N}\right) = (N, A, T, \tau, u),$$

where

- $N = \{1, ..., n\}$ is the player set,
- A_i is the action set for player $i \in N$ with Cartesian product $A = \bigtimes_{i\in N} A_i$ and elements a_i and a, respectively,
- $T = (T_i)_{i\in N}$ is the tuple of type sets T_i for players $i \in N$,
- τ is the probability distribution on T, and
- $u_i : A \times T \to \mathbb{R}$ is player i's payoff function (often $A \times T_i \to \mathbb{R}$)

Ex ante, before the players learn their own types, their beliefs are summarized by τ. The (a priori) probability for type t_i is given by

$$\tau(t_i) := \sum_{t_{-i} \in T_{-i}} \tau(t_{-i}, t_i).$$

Definition (Belief)

Let Γ be a static Bayesian game with probability distribution τ on T. Player i's ex-post (posterior) belief τ_i is the probability distribution on T_{-i} given by the conditional probability

$$\tau_i(t_{-i}) := \tau(t_{-i} | t_i) = \frac{\tau(t_{-i}, t_i)}{\tau(t_i)} = \frac{\tau(t_{-i}, t_i)}{\sum_{t_{-i} \in T_{-i}} \tau(t_{-i}, t_i)}. \tag{1}$$

Problem

Two bidders 1 *and* 2 *with* $T_1 = T_2 = \{high, low\}$ *(willingness to pay) and*

$$\begin{aligned} \tau(high, high) &= \frac{1}{3}, \ \tau(high, low) = \frac{1}{3}, \\ \tau(low, high) &= \frac{1}{9}, \ \tau(low, low) = \frac{2}{9}. \end{aligned}$$

Find

- $\tau(t_2 = high)$ *(ex ante) and*
- $\tau_1(t_2 = high)$ *if player* 1 *has learned that his own willingness to pay is high (ex post).*

Definition

Let Γ be a static Bayesian game.

- A strategy for player $i \in N$ is a function $s_i : T_i \to A_i$. We sometimes write $s(t)$ instead of $(s_1(t_1), ..., s_n(t_n)) \in A$.
- A strategy combination $s^* = (s_1^*, s_2^*, ..., s_n^*)$ is a Bayesian equilibrium (ex post) if

$$s_i^*(t_i) \in \arg\max_{a_i \in A_i} \sum_{t_{-i} \in T_{-i}} \tau_i(t_{-i})\, u_i(a_i, s_{-i}^*(t_{-i}), t_i, t_{-i})$$

holds for all $i \in N$ and all $t_i \in T_i$ obeying $\tau(t_i) > 0$.

The qualification $\tau(t_i) > 0$ is necessary because $\tau_i(t_{-i})$ is ill-defined otherwise. $\tau(t_i) = 0$ implies "anything goes".

Static Bayesian game $\Gamma = (N, (A_1, A_2), (T_1, T_2), \tau, (u_1, u_2))$ with

- the set of two firms $N = \{1, 2\}$,
- the action sets $A_1 = A_2 = [0, \infty)$,
- the type sets $T_1 = \left\{c_1^l, c_1^h\right\} = \{15, 25\}$, $T_2 = \{20\}$,
  ($c_2 = 20$ known to both, c_1 known to 1, only)
- probability distribution τ on T given by $\tau(15, 20) = \tau(25, 20) = \frac{1}{2}$
- inverse demand function given by $p(X) = 80 - X$ and hence
- the payoff functions
  $u_2(x_1, x_2, t_2) = (p(X) - c_2)\,x_2 = (80 - (x_1 + x_2) - 20)\,x_2$ and

$$u_1(x_1, x_2, t_1) = \begin{cases} (80 - (x_1 + x_2) - 15)\,x_1, & t_1 = c_1^l \\ (80 - (x_1 + x_2) - 25)\,x_1, & t_1 = c_1^h \end{cases}$$

The types are independent (trivial).

- Ex-ante probabilities:

$$\begin{aligned}
\tau(20) &:= \tau(15,20) + \tau(25,20) = \frac{1}{2} + \frac{1}{2} = 1, \\
\tau(15) &:= \tau(15,20) = \frac{1}{2}, \text{ and} \\
\tau(25) &:= \tau(25,20) = \frac{1}{2}.
\end{aligned}$$

- Ex-post probability for $c_2 = 20$ (player 1's belief):

$$\tau_1(20) = \frac{\tau(t_1, 20)}{\tau(t_1)} = \frac{\frac{1}{2}}{\frac{1}{2}} = 1 = \tau(20).$$

- Ex-post probability for $c_1 = 15$ (player 2's belief):

$$\tau_2(15) = \frac{\tau(15,20)}{\tau(20)} = \frac{\frac{1}{2}}{1} = \frac{1}{2} = \tau(15).$$

- Strategy sets:
  - player 2: $S_2 = \{s_2 : \{c_2\} \to [0, \infty)\}$ to be identified with A_2
  - player 1 : $S_1 = \left\{ s_1 : \left\{ c_1^l, c_1^h \right\} \to [0, \infty) \right\}$.
- Firm 1's choice depends on its type:

$$
\begin{aligned}
s_1^R(t_1) &= \begin{cases} \arg\max_{x_1 \in [0,\infty)} \left(80 - (x_1 + x_2) - c_1^l\right) x_1, & t_1 = c_1^l \\ \arg\max_{x_1 \in [0,\infty)} \left(80 - (x_1 + x_2) - c_1^h\right) x_1, & t_1 = c_1^h \end{cases} \\
&= \begin{cases} \frac{65}{2} - \frac{1}{2} x_2, & t_1 = c_1^l \\ \frac{55}{2} - \frac{1}{2} x_2, & t_1 = c_1^h \end{cases}
\end{aligned}
$$

- Firm 2's profit is the expected value

$$\begin{aligned} & \tau(15)\left(80-\left[x_1^l+x_2\right]-20\right)x_2 \\ & +\tau(25)\left(80-\left[x_1^h+x_2\right]-20\right)x_2 \\ = & \left(60-\frac{1}{2}\left[x_1^l+x_1^h\right]\right)x_2-x_2^2 \end{aligned}$$

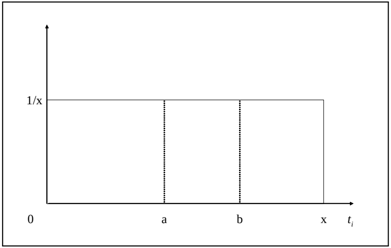
which leads to the reaction function

$$x_2^R=s_2^R(20)=30-\frac{1}{4}\left[x_1^l+x_1^h\right].$$

- Three equations in the three unknowns x_2, x_1^l, and x_1^h.
- They lead to the Nash equilibrium

$$x_2^*=20, s_1^*\left(c_1^l\right)=\frac{45}{2}, \text{ and } s_1^*\left(c_1^h\right)=\frac{35}{2}.$$

- We assume two players $i = 1, 2$ with types t_i from $T_i = [0, x]$, $x > 0$.
- Types are independent.



- Density function:

$$\tau^x(a) = \begin{cases} \frac{1}{x}, & a \in [0, x] \\ 0, & a \notin [0, x] \end{cases}$$

- Hence, for $0 \leq a \leq b \leq x$,

$$\tau^x([a, b]) = \int_a^b \tau^x(t)\, dt = \frac{b - a}{x}.$$

- The probability for a specific type a is zero: $\tau^x([a, a]) = \frac{a-a}{x} = 0$.
- Careful: Distinguish $\tau^x([a, a])$ from $\tau^x(a)$.

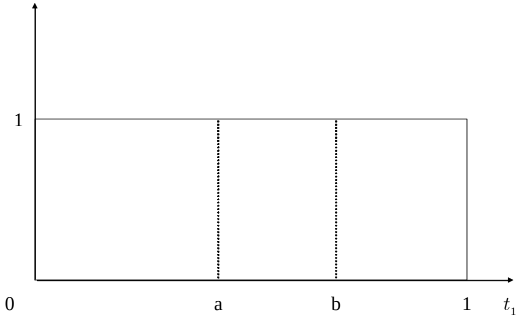
- Static Bayesian games allow a fresh look on mixed equilibria.
- For a given matrix game with a mixed-strategy equilibrium, we construct a sequence of static Bayesian games that converges towards that game.
- In every Bayesian game, no player $i \in N$ randomizes. However, from the point of view of the other players from $N \backslash \{i\}$ who do not know the type t_i, it may well seem as if player i is a randomizer.

		Peter	
		theatre	football
Cathy	theatre	$2+t_C, 1$	$0, 0$
	football	$0, 0$	$1, 2+t_P$

Three equilibria. $\left(\left(\frac{2}{3}, \frac{1}{3}\right), \left(\frac{1}{3}, \frac{2}{3}\right)\right)$ is the properly mixed one for $t_C = t_P = 0$.

Result (as shown in the manuscript): Cathy's probabilities in the static Bayesian games (which depend on x) converge towards her equilibrium mixed strategy as x goes towards 0.

- Two bidders 1 and 2 with types $t_1, t_2 \in [0, 1]$.
- Player 1's distribution is given by



Formally, the first-price auction is the static Bayesian game

$$\Gamma = \left(N, \left(A_1, A_2\right), \left(T_1, T_2\right), \tau, \left(u_1, u_2\right)\right)$$

where

- $N = \{1, 2\}$ is the set of the two bidders,
- $A_1 = A_2 = [0, \infty)$ are the sets of bids chosen by the bidders,
- $T_1 = T_2 = [0, 1]$ are the type sets,
- τ is the a probability distribution on T given by $\tau([a, b], [c, d]) = (b - a)(d - c)$ where $0 \leq a \leq b \leq 1$ and $0 \leq c \leq d \leq 1$ hold, and

- the payoff functions are defined by

$$u_1(a, t_1) = \begin{cases} t_1 - a_1, & a_1 > a_2, \\ \frac{t_1 - a_1}{2}, & a_1 = a_2, \\ 0, & a_1 < a_2, \end{cases} \quad \text{and}$$

$$u_2(a, t_2) = \begin{cases} 0 & a_1 > a_2, \\ \frac{t_2 - a_2}{2} & a_1 = a_2, \\ t_2 - a_2 & a_1 < a_2. \end{cases}$$

In order to solve the first-price auction, we use the ex-post equilibrium definition. For example, if player 1 is of type $t_1 \in [0, 1]$, his condition for the equilibrium strategy combination (s_1^*, s_2^*) is

$$s_1^*(t_1) \in \arg\max_{a_1 \in A_1} \left((t_1 - a_1) \underbrace{\tau(\{t_2 \in [0,1] : a_1 > s_2^*(t_2)\})}_{\substack{\text{probability that player 1's bid} \\ \text{is higher than player 2's bid}}} + \frac{1}{2}(t_1 - a_1) \underbrace{\tau(\{t_2 \in [0,1] : a_1 = s_2^*(t_2)\})}_{\text{probability for equal bids } = 0} \right).$$

- Following Gibbons (1992, pp. 152), we restrict our search for equilibrium strategies to linear strategies of the forms

$$\begin{aligned} s_1^*(t_1) &= c_1 + d_1 t_1 \; (d_1 > 0), \\ s_2^*(t_2) &= c_2 + d_2 t_2 \; (d_2 > 0). \end{aligned}$$

- By

$$\begin{aligned} & \tau(\{t_2 \in [0,1] : a_1 > c_2 + d_2 t_2\}) \\ = & \; \tau\left(\left\{t_2 \in [0,1] : t_2 < \frac{a_1 - c_2}{d_2}\right\}\right) \\ = & \; \tau\left(\left[0, \frac{a_1 - c_2}{d_2}\right]\right) = \frac{a_1 - c_2}{d_2}, \end{aligned}$$

... (next slide)

- player 1's maximization problem is solved by

$$s_1^R(t_1) = \arg\max_{a_1 \in A_1} (t_1 - a_1)\frac{a_1 - c_2}{d_2} = \frac{c_2 + t_1}{2}.$$

- Player 1's best response to c_2 (and hence to player 2's strategy) is a linear strategy with $c_1 = \frac{c_2}{2}$ and $d_1 = \frac{1}{2}$.
- Analogously, bidder 2's best response is

$$s_2^R(t_2) = \frac{c_1 + t_2}{2}$$

with $c_2 = \frac{c_1}{2}$ and $d_2 = \frac{1}{2}$.

- Now, $c_1 = \frac{c_2}{2} = \frac{\frac{c_1}{2}}{2} = \frac{c_1}{4}$ implies $c_1 = 0$. Equilibrium candidate:

$$\begin{aligned} s_1^* &: [0,1] \to \mathbb{R}_+, \quad t_1 \mapsto s_1^*(t_1) = \frac{t_1}{2} \text{ and} \\ s_2^* &: [0,1] \to \mathbb{R}_+, \quad t_2 \mapsto s_2^*(t_2) = \frac{t_2}{2} \end{aligned}$$

The first-price auction

Solution

- These strategies form an equilibrium because the strategies are best reponses to each other. If player 2 uses the (linear) strategy s_2^*, $s_1^*(t_1) = \frac{t_1}{2}$ is a best response as shown above. Thus, s_1^* turns out to be a linear strategy.
- Therefore, we have found an equilibrium in linear strategies but cannot exclude the possibility of an equilibrium in non-linear strategies.

The first-price auction

First-price or second-price auction?

- We now take the auctioneer's perspective and ask the question whether the first-price auction is preferable to the second-price auction. The auctioneer compares the prices
  - $\min(t_1, t_2)$ for the second-price auction and
  - $\max\left(\frac{1}{2}t_1, \frac{1}{2}t_2\right)$ for the first-price auction.
- We assume a risk-neutral auctioneer who maximizes the expected payoff.
- It can be shown (see manuscript) that the auctioneer is indifferent between the first-price and the second-price auction!

Further exercise

Game G_A with probability $p > \frac{1}{2}$, game G_B with probability $1 - p$. Assume $L > M > 1$.

G_A	left	right
up	M , M	1 , $-L$
down	$-L$, 1	0 , 0

G_B	left	right
up	0 , 0	1 , $-L$
down	$-L$, 1	M , M

(a) Assume that both players are informed which game they play before they choose their actions. Formulate this game as a static Bayesian game!

(b*) Assume that player 1 learns whether they play G_A or G_B while player 2 does not. Formulate this game as a static Bayesian game and determine all of its equilibria!