Advanced Microeconomics

Pareto optimality in microeconomics

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Part D. Bargaining theory and Pareto optimality

1. **Pareto optimality in microeconomics**
2. Cooperative game theory

Pareto optimality in microeconomics

overview

1. Introduction: Pareto improvements
2. Identical marginal rates of substitution
3. Identical marginal rates of transformation
4. Equality between marginal rate of substitution and marginal rate of transformation

Pareto optimality in microeconomics

overview

- **Introduction: Pareto improvements**
- Identical marginal rates of substitution
- Identical marginal rates of transformation
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Introduction: Pareto improvements

- Judgements of economic situations
- Ordinal utility $\leftrightsquigarrow$ comparison among different people
- Vilfredo Pareto, Italian sociologist, 1848-1923:

Definition

- Situation 1 is called Pareto superior to situation 2 (a Pareto improvement over situation 2) if no individual is worse off in the first than in the second while at least one individual is strictly better off.
- Situations are called Pareto efficient, Pareto optimal or just efficient if Pareto improvements are not possible.

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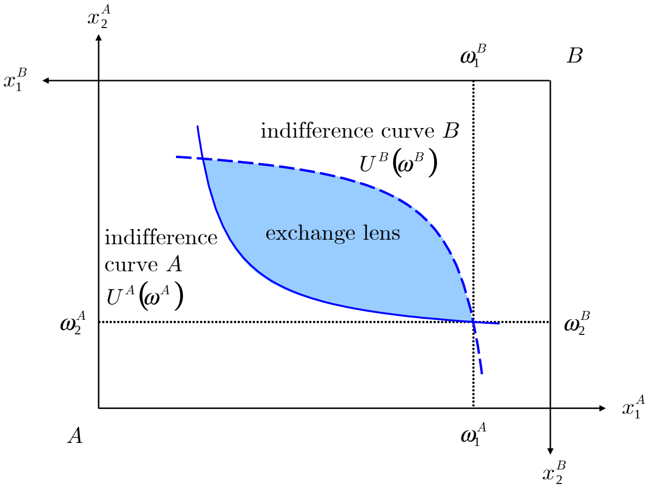
MRS = MRS

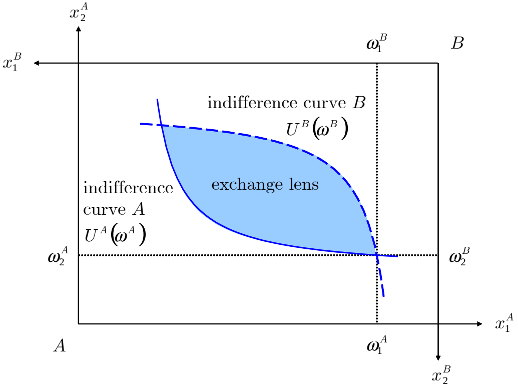
The Edgeworth box for two consumers

Francis Ysidro Edgeworth (1845-1926): "Mathematical Psychics"



MRS = MRS

The Edgeworth box for two consumers



Problem

Which bundles of goods does individual A prefer to his endowment?

MRS = MRS

The Edgeworth box for two consumers

- Consider

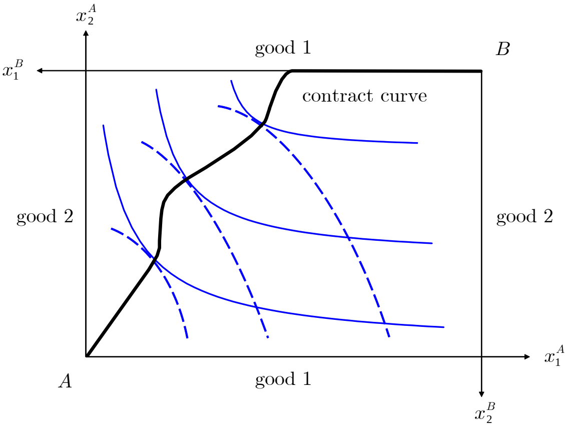
$$(3 =) \left| \frac{dx_2^A}{dx_1^A} \right| = MRS^A < MRS^B = \left| \frac{dx_2^B}{dx_1^B} \right| (= 5)$$

- If A gives up a small amount of good 1,
- he needs MRS^A units of good 2 in order to stay on his indifference curve.
- If individual B obtains a small amount of good 1,
- she is prepared to give up MRS^B units of good 2.
- $\frac{MRS^A + MRS^B}{2}$ units of good 2 given to A by B leave both better off
- Ergo: Pareto optimality requires $MRS^A = MRS^B$

MRS = MRS

The Edgeworth box for two consumers

Pareto optima in the Edgeworth box
– contract curve aka exchange curve



MRS = MRS

The Edgeworth box for two consumers

Problem

Two consumers meet on an exchange market with two goods. Both have the utility function $U(x_1, x_2) = x_1 x_2$*. Consumer A's endowment is* $(10, 90)$*, consumer B's is* $(90, 10)$*.*

a) Depict the endowments in the Edgeworth box!

b) Find the contract curve and draw it!

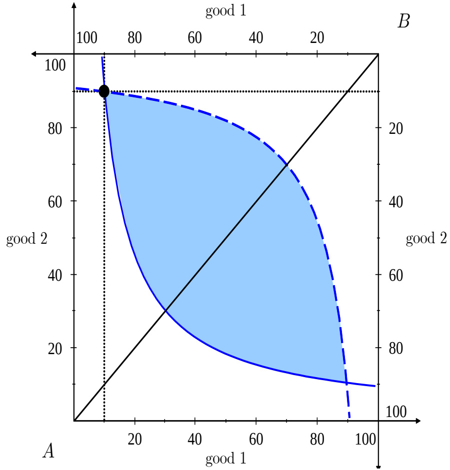
c) Find the best bundle that consumer B can achieve through exchange!

d) Draw the Pareto improvement (exchange lens) and the Pareto-efficient Pareto improvements!

MRS = MRS

The Edgeworth box for two consumers

a)



Solution

b) $x_1^A = x_2^A$,

c) $\left(x_1^B, x_2^B\right) = (70, 70)$.

d) The exchange lens is dotted. The Pareto efficient Pareto improvements are represented by the contract curve within this lens.

Exchange Edgeworth box

the generalized Edgeworth box

Generalization

- n households, $i \in N := \{1, 2, .., n\}$
- ℓ goods, $g = 1, ..., \ell$
- ω_g^i – i's endowment of good g
- $\omega^i := \left(\omega_1^i, ..., \omega_\ell^i\right)$ and $\omega_g := \left(\omega_g^1, ..., \omega_g^n\right)$
- $\sum_{i=1}^n \omega^i \neq \sum_{g=1}^\ell \omega_g$

Problem

Consider two goods and three households and explain ω^3, ω_1 and ω.

Exchange Edgeworth box

the generalized Edgeworth box

Definition

- Functions $N \longrightarrow \mathbb{R}^\ell_+$, i.e. vectors $(x^i)_{i=1,\ldots,n}$ or $\left(x^i\right)_{i \in N}$ where x^i is a bundle from $\mathbb{R}^\ell_+$– allocations.
- Feasible allocations fulfill

$$\sum_{i=1}^{n} x^i \leq \sum_{i=1}^{n} \omega^i$$

MR(T)S = MR(T)S

The production Edgeworth box for two products

- Analogous to exchange Edgeworth box
- $MRTS_1 = \left| \frac{dC_1}{dL_1} \right|$
- Pareto efficiency

$$\left| \frac{dC_1}{dL_1} \right| = MRTS_1 \stackrel{!}{=} MRTS_2 = \left| \frac{dC_2}{dL_2} \right|$$

MRS = MRS

Two markets – one factory

- A firm that produces in one factory but supplies two markets 1 and 2.
- Marginal revenue $MR = \frac{dR}{dy_i}$ can be seen as the monetary marginal willingness to pay for selling one extra unit of good i.
  - Denominator good —> good 1 or 2, respectively
  - Nominator good —> "money" (revenue).
- Profit maximization by a firm selling on two markets 1 and 2 implies

$$\left|\frac{dR}{dy_1}\right| = MR_1 \overset{!}{=} MR_2 = \left|\frac{dR}{dy_2}\right|$$

MRS = MRS

Two firms in a cartel

- The monetary marginal willingness to pay for producing *and* selling one extra unit of good y is a marginal rate of substitution.
- Two firms in a cartel maximize

$$\Pi_{1,2}\left(y_1, y_2\right) = \Pi_1\left(y_1, y_2\right) + \Pi_2\left(y_1, y_2\right)$$

  with FOCs

$$\frac{\partial \Pi_{1,2}}{\partial y_1} \overset{!}{=} 0 \overset{!}{=} \frac{\partial \Pi_{1,2}}{\partial y_2}$$

- If $\frac{\partial \Pi_{1,2}}{\partial y_2}$ were higher than $\frac{\partial \Pi_{1,2}}{\partial y_1}$ ...

How about the Cournot duopoly with FOCs

$$\frac{\partial \Pi_1}{\partial y_1} \overset{!}{=} 0 \overset{!}{=} \frac{\partial \Pi_2}{\partial y_2}?$$

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MRT = MRT

Two factories – one market

- Marginal cost $MC = \frac{dC}{dy}$ is a monetary marginal opportunity cost of production

$$MRT = \left| \frac{dy_2}{dy_1} \right|^{\text{transformation curve}}$$

- One firm with two factories or a cartel in case of homogeneous goods:

$$MC_1 \stackrel{!}{=} MC_2.$$

- Pareto improvements (optimality) have to be defined relative to a specific group of agents!

David Ricardo (1772–1823)
"comparative cost advantage", for example

$$4 = MRT^P = \left| \frac{dW}{dCl} \right|^P > \left| \frac{dW}{dCl} \right|^E = MRT^E = 2$$

Lemma

Assume that f is a differentiable transformation function $y_1 \mapsto y_2$. Assume also that the cost function $C(y_1, y_2)$ is differentiable. Then, the marginal rate of transformation between good 1 and good 2 can be obtained by

$$MRT = \left| \frac{df(y_1)}{dy_1} \right| = \frac{MC_1}{MC_2}.$$

MRT = MRT

Proof.

- Assume a given volume of factor endowments and given factor prices. Then, the overall cost for the production of goods 1 and 2 are constant along the transformation curve:

$$C\left(y_1, y_2\right) = C\left(y_1, f\left(y_1\right)\right) = \text{constant}.$$

- Forming the derivative yields

$$\frac{\partial C}{\partial y_1} + \frac{\partial C}{\partial y_2}\frac{df\left(y_1\right)}{dy_1} = 0.$$

- Solving for the marginal rate of transformation yields

$$MRT = -\frac{df\left(y_1\right)}{dy_1} = \frac{MC_1}{MC_2}.$$

MRT = MRT

International trade

- Before Ricardo:
  England exports cloth and imports wine if

$$\begin{aligned} MC^E_{Cl} &< MC^P_{Cl} \text{ and} \\ MC^E_W &> MC^P_W \end{aligned}$$

  hold.
- Ricardo:

$$\frac{MC^E_{Cl}}{MC^E_W} < \frac{MC^P_{Cl}}{MC^P_W}$$

  suffices for profitable international trade.

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Base case

- Assume

$$MRS = \left|\frac{dy_2}{dy_1}\right|^{\text{indifference curve}} < \left|\frac{dy_2}{dy_1}\right|^{\text{transformation curve}} = MRT$$

- If the producer reduces the production of good 1 by one unit ...
- Inequality points to a Pareto-inefficient situation
- Pareto-efficiency requires

$$MRS \stackrel{!}{=} MRT$$

MRS = MRT

Perfect competition - output space

- FOC output space

$$p \overset{!}{=} MC$$

- Let good 2 be money with price 1
- *MRS* is
  - consumer's monetary marginal willingness to pay for one additional unit of good 1
  - equal to p for marginal consumer
- *MRT* is the amount of money one has to forgo for producing one additional unit of good 1, i.e., the marginal cost
- Thus,

$$\text{price} = \text{marginal willingness to pay} \overset{!}{=} \text{marginal cost}$$

which is also fulfilled by first-degree price discrimination.

MRS = MRT

Perfect competition - input space

FOC input space

$$MVP = p\frac{dy}{dx} \overset{!}{=} w$$

where

- the marginal value product MVP is the monetary marginal willingness to pay for the factor use and
- w, the factor price, is the monetary marginal opportunity cost of employing the factor.

MRS = MRT

Cournot monopoly

For the Cournot monopolist, the $MRS \overset{!}{=} MRT$ can be rephrased as the equality between

- the monetary marginal willingness to pay for selling – this is the marginal revenue $MR = \frac{dR}{dy}$ – and
- the monetary marginal opportunity cost of production, the marginal cost $MC = \frac{dC}{dy}$

MRS = MRT

Household optimum

Consuming household "produces" goods by using his income to buy them, $m = p_1 x_1 + p_2 x_2$, which can be expressed with the transformation function

$$x_2 = f(x_1) = \frac{m}{p_2} - \frac{p_1}{p_2} x_1 .$$

Hence,

$$MRS \overset{!}{=} MRT = MOC = \frac{p_1}{p_2}$$

Sum of MRS = MRT

Public goods

- Definition: non-rivalry in consumption
- Setup:
  - A and B consume a private good x (x^A and x^B)
  - and a public good G
- Optimality condition

$$
\begin{aligned}
& MRS^A + MRS^B \\
= \quad & \left|\frac{dx^A}{dG}\right|^{\text{indifference curve}} + \left|\frac{dx^B}{dG}\right|^{\text{indifference curve}} \\
\stackrel{!}{=} & \left|\frac{d\left(x^A + x^B\right)}{dG}\right|^{\text{transformation curve}} = MRT
\end{aligned}
$$

- Assume $MRS^A + MRS^B < MRT$. Produce one additional unit of the public good ...

Sum of MRS = MRT

Public goods

- Good x as the numéraire good (money with price 1)
- Then, the optimality condition simplifies: sum of the marginal willingness' to pay equals the marginal cost of the good.

Sum of MRS = MRT

Public goods

Problem

In a small town, there live 200 people $i = 1, ..., 200$ with identical preferences. Person i's utility function is $U_i(x_i, G) = x_i + \sqrt{G}$, where x_i is the quantity of the private good and G the quantity of the public good. The prices are $p_x = 1$ and $p_G = 10$, respectively. Find the Pareto-optimal quantity of the public good.

Solution

- $MRT = \left| \frac{d\left(\sum_{i=1}^{200} x_i\right)}{dG} \right|$ *equals* $\frac{p_G}{p_x} = \frac{10}{1} = 10.$
- *MRS for inhabitant i is* $\left| \frac{dx^i}{dG} \right|^{\text{indifference curve}} = \frac{MU_G}{MU_{x^i}} = \frac{\frac{1}{2\sqrt{G}}}{1} = \frac{1}{2\sqrt{G}}.$
- *Hence:* $200 \cdot \frac{1}{2\sqrt{G}} \overset{!}{=} 10$ *and* $G = 100.$

Further exercises: Problem 1

Agent A has preferences on (x_1, x_2), that can be represented by $u^A(x_1^A, x_2^A) = x_1^A$. Agent B has preferences, which are represented by the utility function $u^B(x_1^B, x_2^B) = x_2^B$. Agent A starts with $\omega_1^A = \omega_2^A = 5$, and B has the initial endowment $\omega_1^B = 4, \omega_2^B = 6$.

(a) Draw the Edgeworth box, including

- ω,
- an indifference curve for each agent through ω!

(b) Is $(x_1^A, x_2^A, x_1^B, x_2^B) = (6, 0, 3, 11)$ a Pareto-improvement compared to the initial allocation?

(c) Find the contract curve!

Further exercises: Problem 2

Consider the player set $N = \{1, ..., n\}$. Player $i \in N$ has 24 hours to spend on leisure or work, $24 = l_i + t_i$ where l_i denotes i's leisure time and t_i the number of hours that i contributes to the production of a good that is equally distributed among the group. In particular, we assume the utility functions $u_i(t_1, ..., t_n) = l_i + \frac{1}{n} \sum_{j \in N} \lambda t_j, i \in N$. Assume $1 < \lambda$ and $\lambda < n$.

(a) Find the Nash equilibrium!

(b) Is the Nash equilibrium Pareto-efficient?