Advanced Microeconomics

Repeated games

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Part C. Games and industrial organization

1. Games in strategic form
2. Price and quantity competition
3. Games in extensive form
4. **Repeated games**

1. **Example: Repeating the pricing game**
2. Definitions
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Example: Repeating the pricing game I

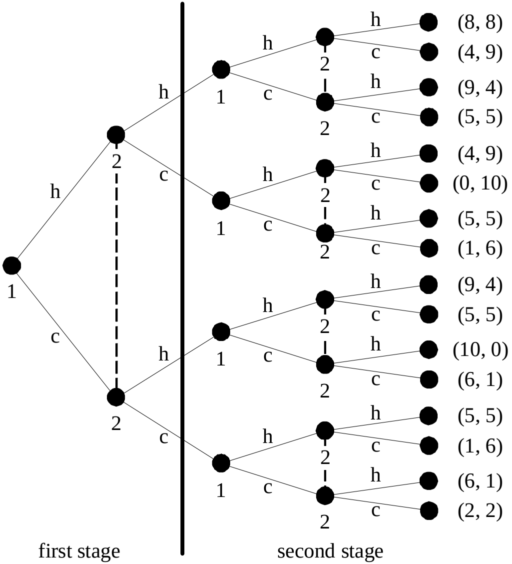
		firm 2	
		high price	cut price
firm 1	high price	$4, 4$	$0, 5$
	cut price	$5, 0$	$1, 1$

Can the twofold repetition of this game help firms to coordinate on the high price?

Problem

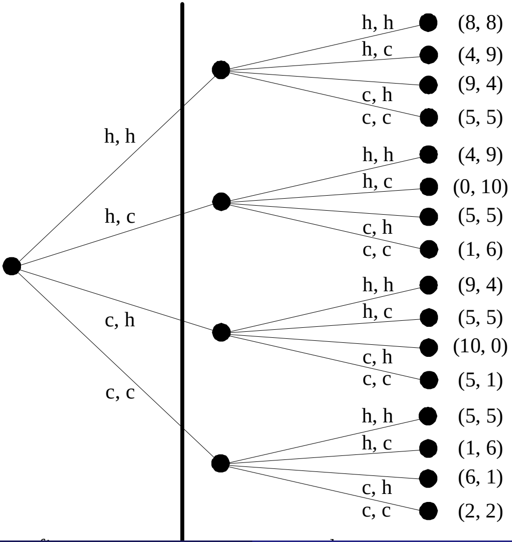
Sketch the very compact form of the twofold repetition!

Example: Repeating the pricing game II



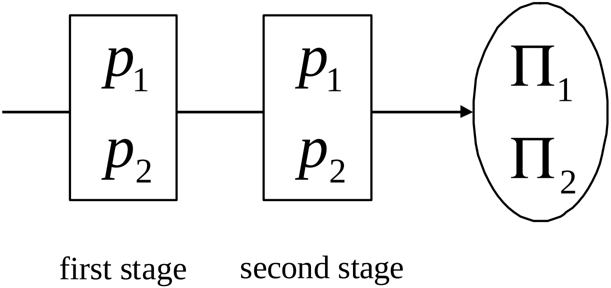
Example: Repeating the pricing game III

multi-stage game:



Example: Repeating the pricing game IV

very compact form:



Example: Repeating the pricing game V

Tit for tat: Begin cooperatively and copy.

Problem

Describe player 2's tit-for-tat strategy as a quintuple of the form

$$\Big\lfloor \underbrace{c}_{\substack{\text{Action at}\\ \text{the first stage}}}, \underbrace{h}_{\substack{\text{Action at}\\ \text{the second stage}\\ \text{after } h \text{ by both}}}, \underbrace{c}_{\substack{\text{Action at}\\ \text{the second stage}\\ \text{after } (h,\, c)}}, \underbrace{h}_{\substack{\text{Action at}\\ \text{the second stage}\\ \text{after } (c,\, h)}}, \underbrace{c}_{\substack{\text{Action at}\\ \text{the second stage}\\ \text{after } (c,\, c)}} \Big\rfloor!$$

Equilibrium?

Example: Repeating the pricing game VI

Problem

Defining the order of subgames as in the previous exercise, are the strategy combinations

- $(\lfloor c, c, c, c, c \rfloor, \lfloor c, c, c, c, c \rfloor)$ *or*
- $(\lfloor c, h, h, h, c \rfloor, \lfloor c, h, h, h, c \rfloor)$

Nash equilibria of the twice repeated pricing game?

Finite repetitions versus infinite repetitions!

Definition: finitely repeated game

Given a strategic game $\Gamma = \left(N, (A_i)_{i \in N}, g\right)$,
the finitely repeated game $\Gamma^{t,\delta}$, $t \geq 1$, is the t-stage game where

- at each stage every player $i \in N$ chooses an action from A_i,
- a strategy for player i is a function $s_i : D \to A_i$ where D is the set of stage nodes that depend on the actions chosen at previous stages and
- payoffs are given by

$$u_i(s) = \frac{\sum_{\tau=0}^{t-1} \delta^\tau g_i(s(d_\tau))}{\sum_{\tau=0}^{t-1} \delta^\tau}$$

where

- d_0 is the initial node,
- d_1 the node resulting from the action combination $s(d_0)$
- ...
- d_{t-1} is the node resulting from action combinations $s(d_0)$ through $s(d_{t-2})$

Definition: notation

	stage game Γ	multi-stage game $\Gamma^{t,\delta}$
A_i	strategy set for player i	action set for player i
s_i	-	strategy for player i
g	payoff function for game	payoff function for stage
u	-	payoff function for game

	stage game	multi-stage game
strategy set for player i	A_i	S_i
payoff function	g	u

Definition: explanation

- $0 \leq \delta \leq 1$, $\delta = \frac{1}{1+r}$ discount rate
  - extreme impatience: $\delta = 0 \longrightarrow u_i(s) = g_i(s(d_0))$ (because of $0^0 = 1$)
  - extreme patience: $\delta = 1 \longrightarrow u_i(s) = \frac{1}{t}\sum_{\tau=0}^{t-1} g_i(s(d_\tau))$
- Denominator $\sum_{\tau=0}^{t-1} \delta^\tau$
  - is a normalization that allows a comparison of multi-stage payoffs with payoffs in the stage game
  - does not influence best responses or equilibria

Problem

Assume that all players use constant strategies, i.e., there is some $a \in A$ such that $s(d) = a \in A$ for all $d \in D$. Find $u_i(s)$.

Definition: infinitely repeated game

Given a strategic game $\Gamma = \left(N, (A_i)_{i \in N}, g\right)$,
the infinitely repeated game $\Gamma^{\infty,\delta}$ is the stage game with infinite stages where

- at each stage every player $i \in N$ chooses an action from A_i,
- a strategy for player i is a function $s_i : D \to A_i$ where D is the set of stage nodes that depend on the actions chosen at previous stages and
- payoffs for $\delta < 1$ are given by

$$
\begin{aligned}
u_i(s) &= \frac{\sum_{\tau=0}^{\infty} \delta^\tau g_i(s(d_\tau))}{\sum_{\tau=0}^{\infty} \delta^\tau} \\
&= (1-\delta) \sum_{\tau=0}^{\infty} \delta^\tau g_i(s(d_\tau)), 0 \leq \delta < 1
\end{aligned}
$$

because of

$$
\sum_{\tau=0}^{\infty} \delta^\tau = \frac{\text{first term}}{1 - \text{factor}} = \frac{\delta^0}{1-\delta} = \frac{1}{1-\delta} \text{ (note } \delta < 1)
$$

Equilibria of stage games and of repeated games I

Theorem

Let $\Gamma = \left(N, (A_i)_{i\in N}, g\right)$ *be a strategic game and let* $\delta \in [0,1)$. *Let* $a^* = (a_1^*, a_2^*, ..., a_n^*) \in A$ *be an equilibrium of* Γ. *Then,* s^* *is a subgame-perfect equilibrium of* Γ^t *or* Γ^∞ *if all the strategies* s_i^* *are constant and equal to* a_i^**, i.e., if for all* $i \in N$ *we have*

$$s_i^* : D \to A_i, \quad d \mapsto s_i^*(d) = a_i^*.$$

Proof.

- Equilibrium: If the other players stick to a_{-i}^* at each stage, the best that player i can do is to choose a_i^* at each stage, also.
- Subgame-perfect equilibrium: Think backward induction!

□

Can you think of a generalization to several equilibria of Γ?

Equilibria of stage games and of repeated games II

The inverse for finite repetitions:

Theorem

Let $\Gamma = \left(N, (A_i)_{i \in N}, g\right)$ *be a strategic game and let* $\delta \in [0,1]$ *and let* s^* *is a subgame-perfect equilibrium of* Γ^t. *Then,* s^* *lets the players choose stage equilibria at each node* $d \in D$.

In particular, any finite repetition of a stage game with only one equilibrium, such as the prisoners' dilemma, has a unique subgame-perfect equilibrium.

		he	
		theatre	football
she	theatre	4, 3	2, 2
	football	1, 1	3, 4

Problem

Show that a finitely repeated game with $\delta = 1$ *may result in an average equilibrium payoff* $3\frac{3}{4}$ *for her.*

Worst punishment I

Definition (Worst punishment)

For $\Gamma = \left(N, (A_i)_{i \in N}, g\right)$,

- the worst punishment inflicted on i : $w_i := \min_{a_{-i}} \max_{a_i} g_i\left(a_i, a_{-i}\right)$
- worst-punishment point: $w = \left(w_1, ..., w_n\right)$
- the worst-punishment action combination(s): $a_{-i}^{pun} := \arg\min_{a_{-i}} \max_{a_i} g_i\left(a_i, a_{-i}\right)$

$$
\begin{aligned}
g_i\left(a_i, a_{-i}\right) &: \quad i\text{'s payoff resulting from } \left(a_i, a_{-i}\right), \\
\max_{a_i} g_i\left(a_i, a_{-i}\right) &: \quad i\text{'s maximal payoff, given } a_{-i}, \\
\min_{a_{-i}} \max_{a_i} g_i\left(a_i, a_{-i}\right) &: \quad i\text{'s minimal (over } a_{-i}\text{) payoff,} \\
\arg\min_{a_{-i}} \max_{a_i} g_i\left(a_i, a_{-i}\right) &: \quad \text{punishing action combination } a_{-i}
\end{aligned}
$$

Worst punishment II

$$w_i = \max_{a_i} g_i\left(a_i, a_{-i}^{pun}\right).$$

Problem

player 2

		left	*right*
player 1	*up*	$2, 1$	$4, 0$
	down	$3, 0$	$1, 1$

Worst punishment for player 1?

Lemma

A player's equilibrium payoff in a finitely or infinitely repeated stage game is not smaller than his worst punishment.

Folk theorems I

			firm 2	
			high price	cut price
firm 1	high price		$4,4$	$0,5$
	cut price		$5,0$	$1,1$

Equilibrium payoffs cannot fall below $w_1 = w_2 = 1$.

Definition

$M \subseteq \mathbb{R}^n$: set of (consumption, payoff, etc.) vectors with $m := |M|$.
hull (M) = convex hull of M =

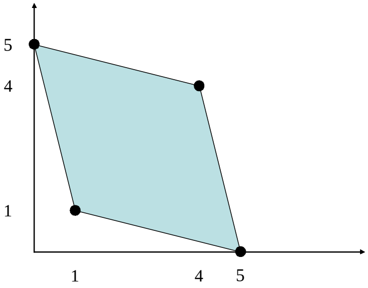
$$\left\{ y \in \mathbb{R}^n : \text{ there exist } x_\ell \in M, \alpha_\ell \geq 0, \text{ and } \sum_{\ell=1}^{m} \alpha_\ell = 1 \text{ s. th. } y = \sum_{\ell=1}^{m} \alpha_\ell x_\ell \right\}$$

Folk theorems II

Definition

- Convex hull of Γ: $hull(\Gamma) := hull(\{g(a) : a \in A\})$.
- Convex hull of Γ above w :
  $hull^w(\Gamma) := hull(\Gamma) \cap \{\pi \in \mathbb{R}^n : \pi_i > w_i\}$

Convex hull of our prisoners' dilemma game:



Convex hull above $(1,1)$ is the subset northeast of $w = (1,1)$.

Folk theorems III

Theorem

Let $\Gamma = \left(N, (A_i)_{i \in N}, g\right)$ *be a strategic game with worst-punishment point* w*. In the infinite repetition* Γ^∞*, every payoff vector from* $hull^w(\Gamma)$ *can be obtained in equilibrium if the discount factor* $\delta < 1$ *is sufficiently large. I.e., for every* $\pi \in hull^w(\Gamma)$*, there is a* $\delta^0 \in (0, 1)$ *such that* $\Gamma^{\infty,\delta}$ *has a Nash equilibrium* s *with* $u(s) = \pi$ *for all* $\delta \in \left(\delta^0, 1\right)$*.*

Idea of a proof:

- Consider a series of action combinations that lead to π.
- Punish players who deviate by subjecting them to the worst punishment forever.
- This punishment can be more severe than the gain obtained by a one-time deviation if δ is sufficiently large.

Details: see manuscript

Further exercises I

Problem 1
Consider the battle of the sexes:

		he	
		theatre	football
she	theatre	4, 3	2, 2
	football	1, 1	3, 4

Can you identify all the equilibria of the twice repeated game? Which of these equilibria are subgame perfect? Write the strategies as quintuples

$$\lfloor a, a_{TT}, a_{TF}, a_{FT}, a_{FF} \rfloor ,$$

where a is the action (theatre or football) at the first stage and a_{TF} the action if she chose theatre at the first stage and he football.

Problem 2

Consider the stage game:

		player 2		
		a_2^1	a_2^2	a_2^3
player 1	a_1^1	$(3,0)$	$(0,1)$	$(0,1)$
	a_1^2	$(0,0)$	$(2,2)$	$(1,0)$
	a_1^3	$(0,0)$	$(0,0)$	$(0,0)$

Is there an equilibrium of the twofold repetition where the two players choose the first actions at the first stage and the second actions at the second stage?

Problem 3

Is there only one equilibrium in a finitely repeated prisoners' dilemma?

		firm 2	
		high price	cut price
firm 1	high price	4, 4	0, 5
	cut price	5, 0	3, 3