Advanced Microeconomics

Games in extensive form

Harald Wiese

University of Leipzig

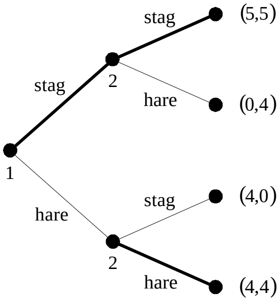
Part C. Games and industrial organization

1. Games in strategic form
2. Price and quantity competition
3. **Games in extensive form**
4. Repeated games

1. **Examples: Non-simultaneous moves in simple bimatrix games**
2. Indian fables
3. Example: the Stackelberg model
4. Transforming an extensive-form game into a strategic-form game
5. Subgame perfection and backward induction
6. Multi-stage games
7. Product differentiation
8. Strategic trade policy

Examples: Non-simultaneous moves in simple bimatrix games

	stag	hare
stag	5, 5 $\boxed{1}\boxed{2}$	0, 4
hare	4, 0	4, 4 $\boxed{1}\boxed{2}$



Player 1's strategies: $\lfloor \text{stag} \rfloor$, $\lfloor \text{hare} \rfloor$
Player 2's strategies: $\lfloor \text{stag, hare} \rfloor$, $\lfloor \text{stag, stag} \rfloor$, $\lfloor \text{hare, hare} \rfloor$, $\lfloor \text{hare, stag} \rfloor$

- backward-induction trails versus
- backward-induction strategy combinations!

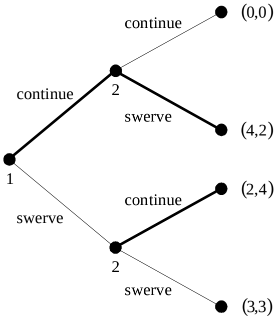
Examples: Non-simultaneous moves in simple bimatrix games

Problem

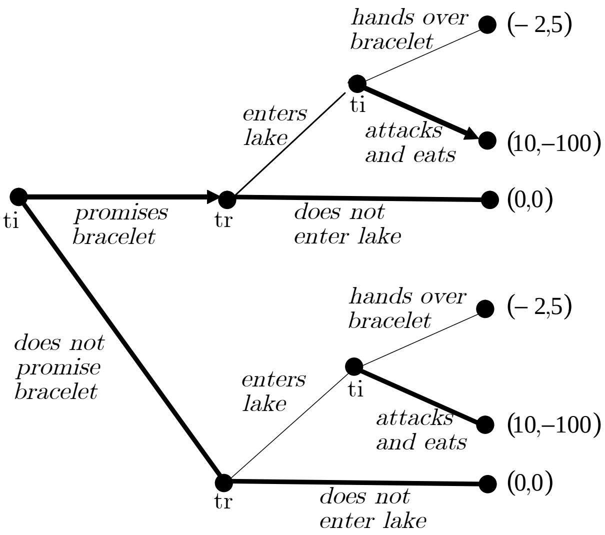
Find the backward-induction solution for the game of chicken!

Solution

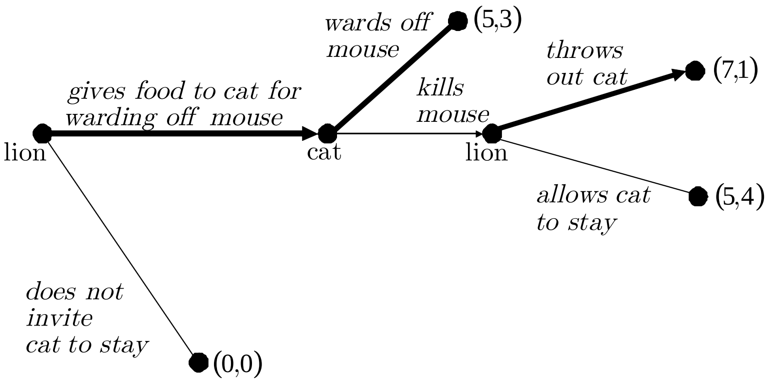
Driver 1 has a first-mover advantage in the game of chicken. He chooses "continue" so that driver 2 is forced to swerve.



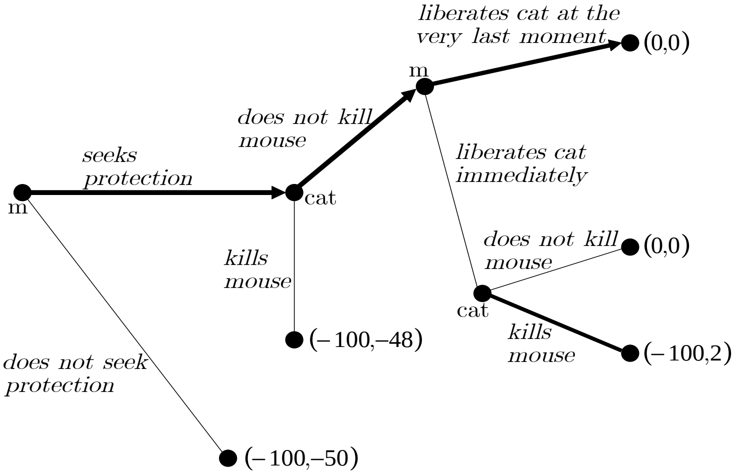
Indian Fables: The tiger and the traveller



Indian Fables: The lion, the mouse, and the cat



Indian Fables: The cat and the mouse



Games in extensive form

overview

1. Examples: Non-simultaneous moves in simple bimatrix games
2. **Example: the Stackelberg model**
3. Transforming an extensive-form game into a strategic-form game
4. Subgame perfection and backward induction
5. Multi-stage games
6. Product differentiation
7. Strategic trade policy

Profit functions:

$$\begin{aligned}\Pi_1(x_1, x_2) &= (a - b(x_1 + x_2))\, x_1 - c_1 x_1 \\ \Pi_2(x_1, x_2) &= (a - b(x_1 + x_2))\, x_2 - c_2 x_2\end{aligned}$$

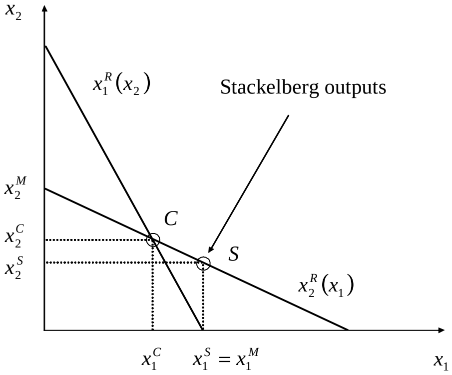
- Leader moves first, x_1
- Follower observes x_1, chooses x_2

$$x_2^R(x_1) = \underset{x_2}{\operatorname{argmax}}\ \Pi_2(x_1, x_2) = \frac{a - c_2}{2b} - \frac{1}{2} x_1$$

- Player 1 anticipates reaction, *reduced profit function*

$$\Pi_1(x_1) := \Pi_1\left(x_1, x_2^R(x_1)\right) = p\left(x_1 + x_2^R(x_1)\right) x_1 - c_1 x_1$$

- Backward-induction quantities: $x_1^S := \arg\max_{x_1} \Pi_1(x_1)$, $x_2^S := x_2^R(x_1^S)$
- Player 1 chooses profit-maximizing point on the follower's reaction curve



Leader's reduced profit function:

$$\begin{aligned}\Pi_1(x_1) &:= \Pi_1\left(x_1, x_2^R(x_1)\right) = p\left(x_1 + x_2^R(x_1)\right)x_1 - c_1 x_1 \\ &= \left(a - b\left(x_1 + \left[\frac{a-c_2}{2b} - \frac{1}{2}x_1\right]\right)\right)x_1 - c_1 x_1 \\ &= ax_1 - bx_1^2 - b\frac{a-c_2}{2b}x_1 + \frac{1}{2}bx_1^2 - c_1 x_1\end{aligned}$$

FOC:

$$MR_1(x_1) = a - bx_1 - \frac{b(a-c_2)}{2b} \overset{!}{=} c_1 = MC_1(x_1)$$

Resulting outputs:

$$\begin{aligned}x_1^S &= \frac{a-2c_1+c_2}{2b}, x_2^S := x_2^R\left(x_1^S\right) = \frac{a+2c_1-3c_2}{4b} \\ X^S &:= x_1^S + x_2^S\end{aligned}$$

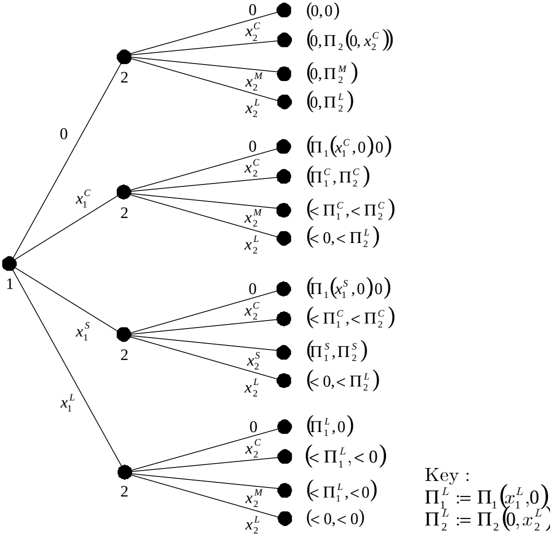
Resulting price:

$$p\left(X^S\right) = a - bX^S = \frac{1}{4}\left(a + 2c_1 + c_2\right)$$

Resulting profits:

$$\Pi_1^S = \frac{1}{8}\frac{\left(a + c_2 - 2c_1\right)^2}{b}, \Pi_2^S = \frac{1}{16}\frac{\left(a - 3c_2 + 2c_1\right)^2}{b}$$

Stackelberg model



- action sets: $[0, \infty)$
- player 2's move depends on player 1's
- $x_2 = s_2(x_1)$

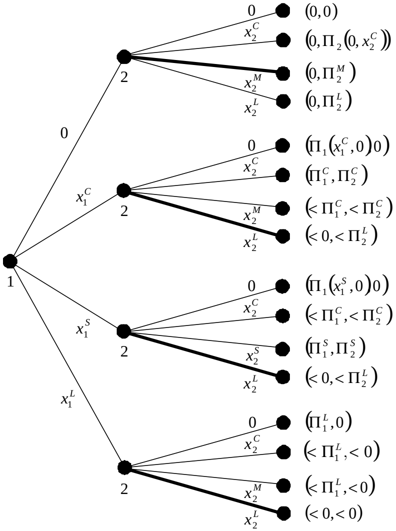
$$\begin{aligned} s_2 \quad : \quad & [0, \infty) \rightarrow [0, \infty), \\ & x_1 \mapsto s_2(x_1) \end{aligned}$$

- e.g.

$$s_2^M : x_1 \mapsto \begin{cases} x_2^M, & x_1 = 0 \\ x_2^L, & x_1 > 0. \end{cases}$$

- Nash equilibrium

$$\left(0, s_2^M\right)$$



- Backward-induction equilibrium

$$\left(x_1^S, x_2^R\right)$$

- Backward-induction quantities: $\left(x_1^S, x_2^R\left(x_1^S\right)\right)$

Key :
$\Pi_1^L := \Pi_1(x_1^L, 0)$
$\Pi_2^L := \Pi_2(0, x_2^L)$

$x_2^S \in \mathbb{R}_+$ can be understood as a constant function $x_2^S : [0, \infty) \longrightarrow [0, \infty)$ with $x_2^S(x_1) = x_2^R\left(x_1^S\right)$ for all $x_1 \in [0, \infty)$

Problem

Which of the following strategy combinations are Nash equilibria of the Stackelberg model?

1. $\left(x_1^S, x_2^R\left(x_1^S\right)\right)$
2. $\left(x_1^S, x_2^R\right)$
3. $\left(x_1^C, x_2^C\right)$

Solution

1. $\left(x_1^S, x_2^R\left(x_1^S\right)\right)$ *is not an equilibrium. Facing* $x_2^S = x_2^R\left(x_1^S\right)$, *firm* 1*'s optimal choice is* $x_1^R\left(x_2^S\right) \neq x_1^S$.
2. $\left(x_1^S, x_2^R\right)$ *is the subgame-perfect equilibrium (obtained by backward induction).*
3. $\left(x_1^C, x_2^C\right)$ *is a Nash equilibrium, but not subgame perfect.*

- Consider $R_1(x_1) = p(X) \cdot x_1$.

$$MR_1(x_1) = p(X) + \frac{dp}{dX}\frac{dX}{dx_1}\, x_1 \qquad \text{(chain rule)}$$

- With $X = x_1 + x_2^R(x_1)$

$$MR_1(x_1) = p(X) + \frac{dp}{dX}\frac{d\left(x_1 + x_2^R(x_1)\right)}{dx_1}\, x_1$$

- Thus $MR_1(x_1) = p(X) + \frac{dp}{dX}\frac{\partial x_1}{\partial x_1}x_1 + \frac{dp}{dX}\frac{\partial x_2^R(x_1)}{\partial x_1}x_1$

$$= \underbrace{p(X) + x_1 \frac{dp(X)}{dX}}_{\text{direct effect}} + \underbrace{x_1 \underbrace{\frac{dp(X)}{dX}}_{<0} \underbrace{\frac{dx_2^R(x_1)}{dx_1}}_{<0}}_{\text{follower effect, } > 0} \qquad \left(\frac{dx_1}{dx_1} = 1\right).$$

- No (positive) follower effect in Cournot's model; $x_1^S > x_1^C$

Stackelberg model

a problem with three firms I

Problem

Three firms, inverse demand $p(X) = 100 - X$. Average cost are zero. Firm 1 moves first; firms 2 and 3 move second and simultaneously.

Solution

- *Firm 2's profit*

$$\begin{aligned} \Pi_2(x_1, x_2, x_3) &= p(X)\, x_2 - C(x_2) \\ &= (100 - x_1 - x_2 - x_3)\, x_2 - C(x_2) \end{aligned}$$

- *Firm 2's and firm 3's reaction functions*

$$\begin{aligned} x_2^R(x_1, x_3) &= \frac{100 - x_1 - x_3}{2} \\ x_3^R(x_1, x_2) &= \frac{100 - x_1 - x_2}{2} \end{aligned}$$

Solution

- *Cournot equilibrium between firms* 2 *and* 3*:* $x_2^C(x_1) = \frac{100-x_1}{3}, x_3^C(x_1) = \frac{100-x_1}{3}$
- *Firm* 1*'s reduced profit*

$$\Pi_1\left(x_1, x_2^C(x_1), x_3^C(x_1)\right) = \left(100 - x_1 - x_2^C(x_1) - x_3^C(x_1)\right)x_1 - 0$$

- $x_1^S = 50$ *and* $x_2^C(50) = x_3^C(50) = \frac{50}{3}$

1. Examples: Non-simultaneous moves in simple bimatrix games
2. Example: the Stackelberg model
3. **Transforming an extensive-form game into a strategic-form game**
4. Subgame perfection and backward induction
5. Multi-stage games
6. Product differentiation

Perfect-information extensive-form game

Game Γ =

- Initial node and exactly one trail
  initial node —> end node
- Player set $N = \{1, ..., n\}$
- Decision nodes D_i, $i \in N$ ($D_i \neq \emptyset$) with union D
  Nodes at which actions can be taken
  $D_i \cap D_j = \emptyset$ for $i \neq j$
- Actions A_d at $d \in D$ and union A
- Terminal nodes = end nodes E :
  with payoff information for all the players
- set of all nodes

$$D \cup E = (D_1 \cup ... \cup D_n) \cup E$$

Strategies

definition

As in extensive-form decision situations, strategies are not given, but need to be defined.

Definition

A strategy for player i is a function $s_i : D_i \to A$ where $s_i(d) \in A_d$ for all $d \in D_i$.

Assuming information partition I_i for player i (imperfect information!):

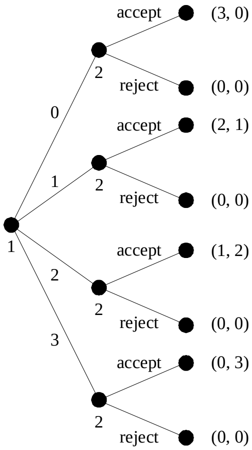
Definition

A strategy for player i is a function $s_i : D_i \to A$ where, for all $d \in D_i$,

- $s_i(d) \in A_d$ and
- $s_i(d) = s_i(d')$ for all $d' \in I_i(d)$.

Transforming extensive- into strategic-form games

example: take it or leave it



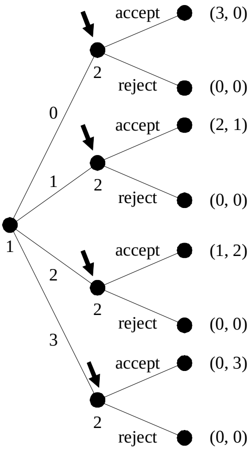
Strategy sets for players 1 and 2?

Transforming extensive- into strategic-form games

example: take it or leave it

Solution

Player 2 has 16 strategies,
e.g. $\lfloor reject, accept, accept, accept \rfloor$,
$\lfloor reject, accept, reject, accept \rfloor$



Player 2's strategies comprise:

- player 2 does not accept:

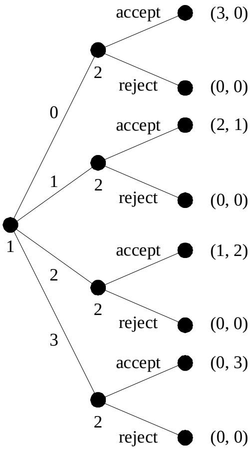
$$\lfloor \text{reject, reject, reject, reject} \rfloor$$

- player 2 accepts iff offered $\geq$ 2 coins:

$$\lfloor \text{reject, reject, accept, accept} \rfloor$$

- player 2 accepts if no coin or two coins are offered to him, otherwise he rejects:

$$\lfloor \text{accept, reject, accept, reject} \rfloor$$



Transforming extensive- into strategic-form games

Once the strategies are defined,
every combination of strategies leads to specific payoffs and
a strategic game Γ is defined.

Best responses & Nash equilibria as usual

Problem

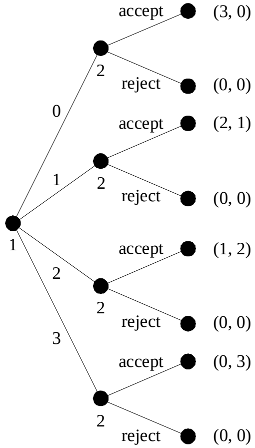
Are

$$\left(\lfloor 2 \rfloor, \lfloor reject,\ reject,\ accept,\ accept \rfloor\right)$$

and

$$\left(\lfloor 2 \rfloor, \lfloor accept,\ reject,\ accept,\ reject \rfloor\right)$$

Nash equilibria?



1. Examples: Non-simultaneous moves in simple bimatrix games
2. Example: the Stackelberg model
3. Transforming an extensive-form game into a strategic-form game
4. **Subgame perfection and backward induction**
5. Multi-stage games
6. Product differentiation
7. Strategic trade policy

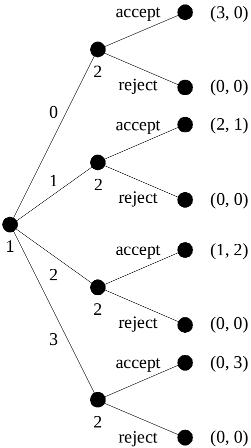
Consider a decision node $w \in D$.
w and the nodes following w make up the set W.

- Game Γ^w generated from game Γ
  $=$ subgame
- if $W \cap D_i \neq \emptyset$,
  strategy s_i^w (in Γ^w) generated from strategy s_i (in Γ)
  $=$ the restriction of s_i to W
  $=$ player i's substrategy of $s_i \in S_i$

Five subgames,
four proper subgames

Strategy
$s_2 = \lfloor \text{accept, reject, accept, reject} \rfloor$
generates substrategies

- s_2 for the improper subgame and
- $\lfloor \text{accept} \rfloor$ or $\lfloor \text{reject} \rfloor$ at the proper subgames



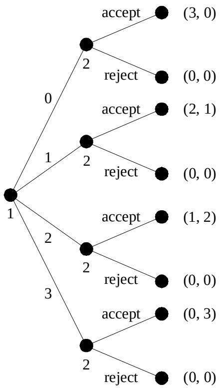
Definition

A strategy combination s is subgame perfect if s^w is a Nash equilibrium in every subgame Γ^w.

Problem

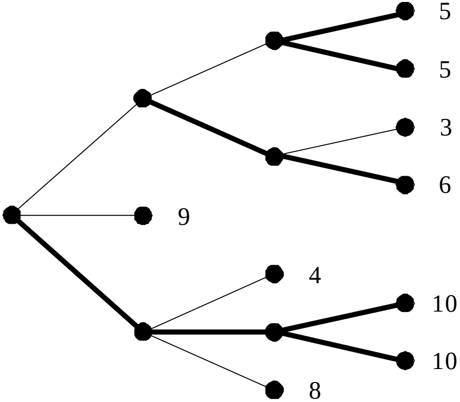
Subgame perfect?

- $(\lfloor 2 \rfloor, \lfloor reject, reject, accept, accept \rfloor)$
- $(\lfloor 0 \rfloor, \lfloor accept, accept, accept, accept \rfloor)$
- $(\lfloor 1 \rfloor, \lfloor reject, accept, accept, accept \rfloor)$
- $(\lfloor 1 \rfloor, \lfloor accept, accept, accept, accept \rfloor)$
- $(\lfloor 0 \rfloor, \lfloor reject, accept, accept, accept \rfloor)$



Backward induction for perfect information

Trivial example: decision situation



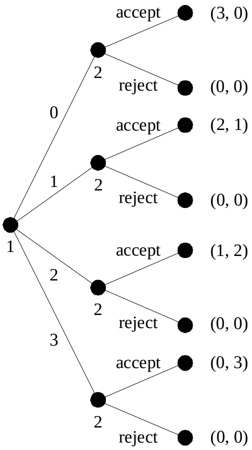
Backward induction means

- starting with the smallest subtrees,
- noting the best actions,
- and working towards the initial node,
- while carrying the payoff information of **all** players

- Backward-induction trails versus
- backward-induction strategy combinations!

Problem

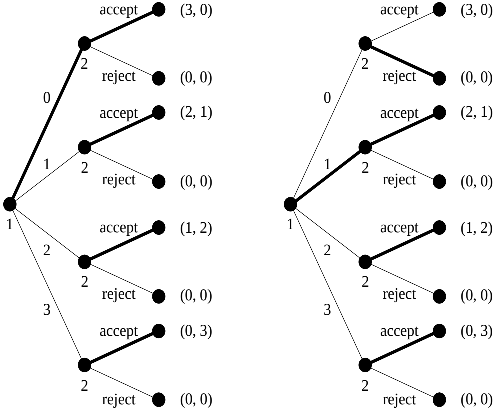
How many backward-induction trails and how many backward-induction strategy combinations can you find?



Solution

Player 2 indifferent when offered 0. Hence

- *two backward-induction trees,*
- *two backward-induction trails and*
- *two backward-induction strategy combinations.*



Subgame perfection and backward induction

Theorem

Let Γ be of finite length. Then,

- *the set of subgame-perfect strategy combinations and*
- *the set of backward-induction strategy combinations*

coincide.

Thus, you can find all subgame-perfect strategy combinations by applying backward induction.

Games in extensive form

overview

1. Examples: Non-simultaneous moves in simple bimatrix games
2. Example: the Stackelberg model
3. Transforming an extensive-form game into a strategic-form game
4. Subgame perfection and backward induction
5. **Multi-stage games**
6. Product differentiation
7. Strategic trade policy

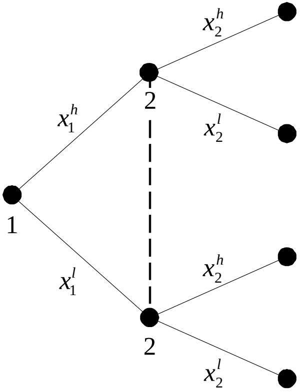
Multi-stage games

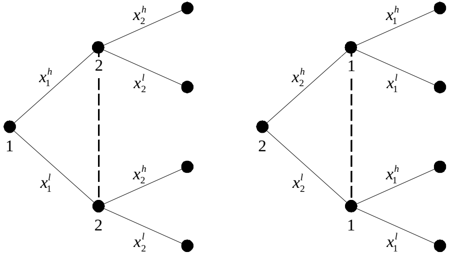
Description

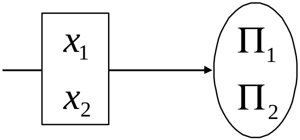
- Every player chooses at most one action at every stage
- Each player knows all the actions undertaken in previous stages but
- no action other than one's own in the present stage

- node $d \in D$ addresses a stage, not a decision node
- strategies assign actions to the stages

Two equivalent Cournot trees,
but how many stages?



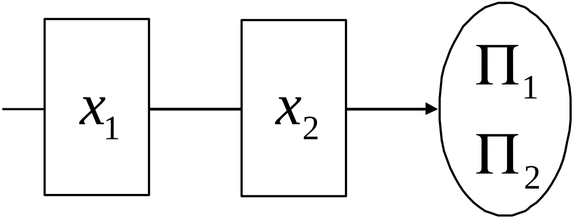




"Very compact form"
of the games
indicated by the trees

One stage, only!

Stackelberg very compact



Problem

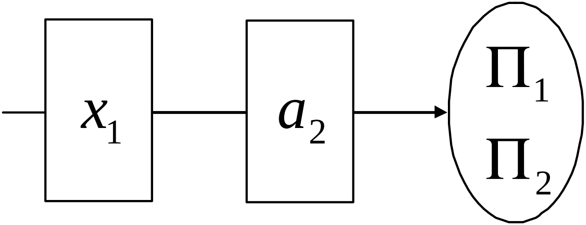
Draw the very compact form of the take-it-or-leave-it game!

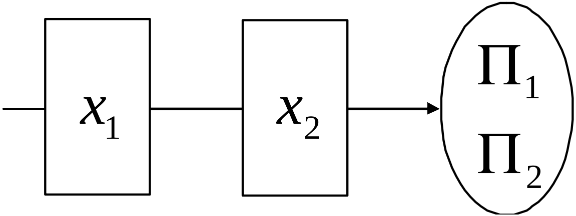
Problem

Draw the very compact form of the take-it-or-leave-it game!

Solution

Player 1 makes an offer x_1 ($x_1 \in \{0,1,2,3\}$) and player 2 gives an answer a_2 ($a_2 \in \{accept, reject\}$) to that offer.





- First focus on last stage

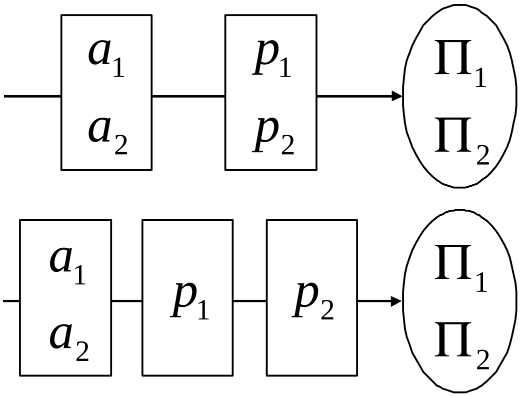
$$\Rightarrow x_2^R$$

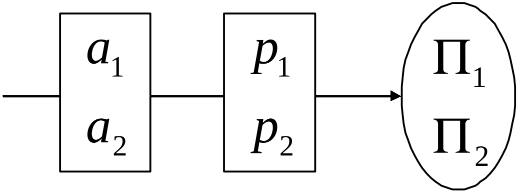
- Substitute into 1's profit function

$$\Rightarrow \text{reduced profit } \Pi_1\left(x_1, x_2^R\left(x_1\right)\right)$$

$\Rightarrow$ Equilibrium quantities: $\left(x_1^S, x_2^R\left(x_1^S\right)\right)$
$\Rightarrow$ Subgame-perfect equilibrium $\left(x_1^S, x_2^R\right)$

Other examples: a_1 and a_2 stand for product varieties, advertising ...





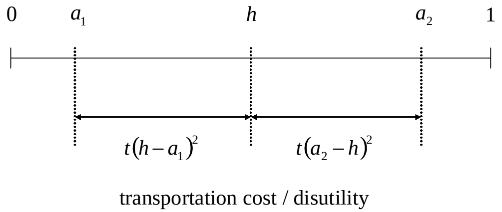
- Last stage: $\left(p_1^B(a_1, a_2), p_2^B(a_1, a_2)\right)$
- Substitute p_1 and p_2 into profit functions $\Rightarrow$ reduced profit functions
- Calculate equilibrium varieties $\left(a_1^N, a_2^N\right)$
- SPE:

$$\left(\left(a_1^N, p_1^B\right), \left(a_2^N, p_2^B\right)\right)$$

  - p_1^B is a function $(a_1, a_2) \mapsto p_1^B(a_1, a_2)$
  - prices in equilibrium: $p_1^B\left(a_1^N, a_2^N\right)$ and $p_2^B\left(a_1^N, a_2^N\right)$

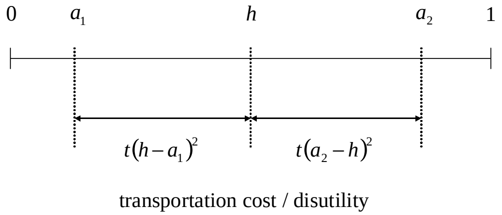
1. Examples: Non-simultaneous moves in simple bimatrix games
2. Example: the Stackelberg model
3. Transforming an extensive-form game into a strategic-form game
4. Subgame perfection and backward induction
5. Multi-stage games
6. **Product differentiation**
7. Strategic trade policy

- Vertical (quality) product differentiation
- Horizontal product differentiation
- Hotelling linear space



Product differentiation

demand functions



Definition

Two products 1 and 2 are homogeneous if $p_1 < p_2$ implies $x_2(p_1, p_2) = 0$ and if $p_1 > p_2$ implies $x_1(p_1, p_2) = 0$.

Products 1 and 2 are homogeneous if $a_1 = a_2$ or $t = 0$ hold as we will see below.

Product differentiation

demand functions

Assumptions:

- Every consumers buys one unit

$$x_1 + x_2 = 1$$

- Consumer at h buys from 1 if

$$p_1 + t\left(h - a_1\right)^2 \leq p_2 + t\left(a_2 - h\right)^2$$

iff

$$h \leq \frac{a_2 + a_1}{2} + \frac{p_2 - p_1}{2t\left(a_2 - a_1\right)} =: h^*$$

Induced demand ($\bar{a} := \frac{a_2+a_1}{2}$ and $\Delta a := a_2 - a_1$):

$$x_1\left(p_1, p_2, a_1, a_2\right) = h^* = \underbrace{\bar{a}}_{\substack{\text{demand} \\ \text{for } p_1 = p_2}} + \underbrace{\frac{1}{2t\Delta a}}_{\substack{\text{competition} \\ \text{intensity}}} \underbrace{\left(p_2 - p_1\right)}_{\substack{\text{firm 1's} \\ \text{price advantage}}}$$

Product differentiation

demand functions

$$x_1(p_1, p_2, a_1, a_2) = h^* = \overline{a} + \frac{1}{2t\Delta a}\ (p_2 - p_1)$$

- *Product differentiation makes demand inelastic* (assume $p_1 = p_2$ and maximal differentiation ($a_1 = 0,\ a_2 = 1$)):

$$\varepsilon_{x_1,p_1}\Big|_{p_1=p_2=p} = \frac{\partial x_1}{\partial p_1}\frac{p_1}{x_1}\Bigg|_{p_1=p_2=p} = \frac{-1}{2t\Delta a}\frac{p_1}{x_1}\Bigg|_{p_1=p_2=p} = -\frac{p}{t}.$$

- *Demand for $p_1 = p_2$ or high differentiation*
  $\Rightarrow$ consumers in $[0, \overline{a})$ buy good 1
- *Product differentiation lessens competition intensity:*

$$\frac{1}{2t\Delta a} = \left|\frac{\partial x_1}{\partial p_1}\right|$$

  Definition: competition intensity high if small changes in variables lead to huge changes of sales or profits

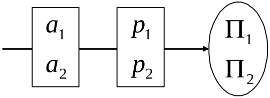
Problem

Assume that the government regulates prices at $p_1 = p_2 > c_1 = c_2$ where c_1 and c_2 are the average costs of the two firms. The firms 1 and 2 simultaneously determine their positions a_1 and a_2, respectively. Can you find an equilibrium?

Four steps:

- In equilibrium, we have $a_1 = a_2$. Otherwise ...
- In equilibrium, we have $a_1 = a_2 = \frac{1}{2}$. Otherwise ...
- $(a_1, a_2) = \left(\frac{1}{2}, \frac{1}{2}\right)$ is an equilibrium. If a firm deviates, ...
- $(a_1, a_2) = \left(\frac{1}{2}, \frac{1}{2}\right)$ is the unique equilibrium.

Remember the political parties in the game-theory chapter!



profit functions

$$\begin{aligned}
\Pi_1 &= (p_1 - c)\,x_1 = (p_1 - c)\left(\bar{a} + \frac{p_2 - p_1}{2t\Delta a}\right) \\
\Pi_2 &= (p_2 - c)\,x_2 = (p_2 - c)\left(1 - \bar{a} + \frac{p_1 - p_2}{2t\Delta a}\right)
\end{aligned}$$

Solving the two-stage game

the second stage

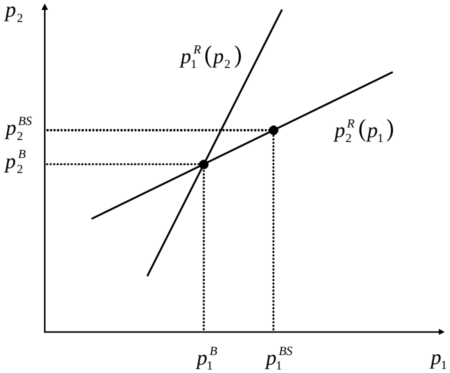
- Backward induction
- $\Pi_1 = (p_1 - c)\left(\overline{a} + \frac{p_2 - p_1}{2t\Delta a}\right), \Pi_2 = (p_2 - c)\left(1 - \overline{a} + \frac{p_1 - p_2}{2t\Delta a}\right)$, disregarding corner solutions:

$$\begin{aligned} p_1^R(p_2) &= \underset{p_1}{\operatorname{argmax}}\ \Pi_1 = \frac{p_2 + c + 2t\overline{a}\Delta a}{2} \\ p_2^R(p_1) &= \underset{p_2}{\operatorname{argmax}}\ \Pi_2 = \frac{p_1 + c + 2t(1-\overline{a})\Delta a}{2} \end{aligned}$$

- Prices are strategic complements. Household theory:
  - demand increase for one good (due to a price decrease)
  - leads to a demand increase of the complement.
- Prices are relatively low if the firms are positioned near each other:

$$\frac{\partial p_1^R(p_2)}{\partial a_1} = -ta_1 \text{ and } \frac{\partial p_1^R(p_2)}{\partial a_2} = ta_2$$

$$p_1^R(p_2) = \frac{p_2 + c + 2t\bar{a}\Delta a}{2}, \qquad p_2^R(p_1) = \frac{p_1 + c + 2t(1-\bar{a})\Delta a}{2}$$



$$\begin{aligned} p_1^B &= c + \frac{2}{3}t(1+\bar{a})\Delta a \\ p_2^B &= c + \frac{2}{3}t(2-\bar{a})\Delta a \end{aligned}$$

Equilibrium quantities: $x_1^B = \frac{1}{3}(1+\bar{a})$, $x_2^B = \frac{1}{3}(2-\bar{a})$

(Reduced) profits: $\Pi_1^B = \dfrac{2}{9}t(1+\bar{a})^2\Delta a$, $\Pi_2^B = \dfrac{2}{9}t(2-\bar{a})^2\Delta a$

$$\Pi_1 = (p_1 - c)\left(\overline{a} + \frac{p_2 - p_1}{2t\Delta a}\right), \Pi_2 = (p_2 - c)\left(1 - \overline{a} + \frac{p_1 - p_2}{2t\Delta a}\right)$$

Problem

Assume maximal differentiation, i.e., $a_1 = 0$ and $a_2 = 1$. Solve the sequential pricing game: firm 1 moves first and firm 2 moves second, just as in the Stackelberg model. Show that we have a second-mover advantage. Show also that the leader's profit is higher in the sequential case than in the simultaneous one. Do you see why this is necessarily true?

Solving the two-stage game

solution I

Solution

- *Firm 2's reaction function*

$$p_2^R(p_1) = \underset{p_2}{\operatorname{argmax}}\, \Pi_2 = \frac{p_1 + c + 2t(1-\bar{a})\Delta a}{2} = \frac{p_1 + c + t}{2}$$

- *Firm 1's reduced profit function*

$$\Pi_1(p_1) = (p_1 - c)\left(\frac{1}{2} + \frac{p_2^R(p_1) - p_1}{2t}\right)$$

- *Equilibrium prices*

$$p_1^{BS} = \underset{p_1}{\operatorname{argmax}}\left(\Pi_1(p_1, p_2^R(p_1))\right) = c + \frac{3t}{2} > c + \frac{5}{4}t = p_2^R\left(p_1^{BS}\right)$$

Solution

- *Second-mover advantage*

$$\begin{aligned}\Pi_1^{BS} &= \left(p_1^{BS}-c\right)\left(\frac{1}{2}+\frac{p_2^R\left(p_1^{BS}\right)-p_1^{BS}}{2t}\right)=\frac{18}{32}t \\ &< \frac{25}{32}t=\left(p_2^R\left(p_1^{BS}\right)-c\right)\left(\frac{1}{2}+\frac{p_1^{BS}-p_2^R\left(p_1^{BS}\right)}{2t}\right)=\Pi_2^{BS}\end{aligned}$$

- *but, of course,*

$$\Pi_1^{BS}=\frac{9}{16}t>\frac{8}{16}t=\Pi_1^B$$

- Reduced profit function

$$\Pi_1^B(a_1, a_2) = \frac{2}{9} t (1 + \bar{a})^2 \Delta a = \frac{1}{18} t (2 + a_1 + a_2)^2 (a_2 - a_1)$$

- Given $0 \leq a_1 \leq a_2$

$$\frac{\partial \Pi_1^B}{\partial a_1} = -\frac{t}{18} (2 + a_1 + a_2)(2 + 3a_1 - a_2) < 0$$

and hence

$$a_1^R(a_2) = 0 \text{ for all } a_2 \geq a_1$$

- Analogously

$$a_2^R(a_1) = 1$$

- First-stage equilibrium

$$\left(a_1^N, a_2^N\right) = (0, 1)$$

Solving the two-stage game
maximal differentiation

- Finally

$$\begin{aligned} p_1^B &= c+t, & p_2^B &= c+t, \\ x_1^B &= \tfrac{1}{2}, & x_2^B &= \tfrac{1}{2}, \\ \Pi_1^B &= \tfrac{1}{2}t, & \Pi_2^B &= \tfrac{1}{2}t. \end{aligned}$$

Direct and strategic effects

accomodation

We evaluate the effect o firm 1 "moving closer" to firm 2 on firm 1's profit. Firm 1's reduced profit:

$$\Pi_1^B(a_1, a_2) = \Pi_1\left(a_1, a_2, p_1^B(a_1, a_2), p_2^B(a_1, a_2)\right)$$

and its derivative with respect to a_1:

$$\frac{\partial \Pi_1^B}{\partial a_1} = \frac{\partial \Pi_1}{\partial a_1} + \frac{\partial \Pi_1}{\partial p_1}\frac{\partial p_1^B}{\partial a_1} + \frac{\partial \Pi_1}{\partial p_2}\frac{\partial p_2^B}{\partial a_1}$$

Direct and strategic effects

accomodation

$$\Pi_1^B(a_1, a_2) = \Pi_1\left(a_1, a_2, p_1^B(a_1, a_2), p_2^B(a_1, a_2)\right)$$

$$\frac{\partial \Pi_1^B}{\partial a_1} = \underbrace{\frac{\partial \Pi_1}{\partial a_1}}_{\substack{>0 \\ \text{direct or} \\ \text{demand effect}}} + \underbrace{\underbrace{\frac{\partial \Pi_1}{\partial p_1}}_{=0}\frac{\partial p_1^B}{\partial a_1}}_{\substack{=0 \\ \text{first-order condition} \\ \text{at stage 2} \\ \text{(envelope theorem)}}} + \underbrace{\underbrace{\frac{\partial \Pi_1}{\partial p_2}}_{>0}\underbrace{\frac{\partial p_2^B}{\partial a_1}}_{<0}}_{\substack{<0 \\ \text{strategic effect} \\ \text{of positioning}}}$$

- Using $\Pi_1 = (p_1 - c)\, x_1$

$$\Pi_1^B(a_1,a_2) = \left(p_1^B(a_1,a_2) - c\right) x_1\left(a_1,a_2,p_1^B(a_1,a_2),p_2^B(a_1,a_2)\right)$$

- $$\frac{\partial \Pi_1^B}{\partial a_1} = \underbrace{\left(p_1^B(a_1,a_2) - c\right)\frac{\partial x_1}{\partial a_1}}_{\substack{>0 \\ \text{direct or} \\ \text{demand effect}}} + \underbrace{\underbrace{\left(p_1^B(a_1,a_2) - c\right)}_{>0}\underbrace{\frac{\partial x_1}{\partial p_2}}_{>0}\underbrace{\frac{\partial p_2^B}{\partial a_1}}_{<0}}_{\substack{<0 \\ \text{strategic effect} \\ \text{of positioning}}}$$

- Quadratic transportation costs imply

$$\frac{\partial \Pi_1^B}{\partial a_1} \leq 0.$$

$$\Pi_2^B(a_1,a_2) = \Pi_2\left(a_2, a_1, p_2^B(a_1,a_2), p_1^B(a_1,a_2)\right)$$

$$\frac{\partial \Pi_2^B}{\partial a_1} = \underbrace{\frac{\partial \Pi_2}{\partial a_1}}_{\substack{<0 \\ \text{direct or} \\ \text{demand effect}}} + \underbrace{\underbrace{\frac{\partial \Pi_2}{\partial p_2}}_{=0}\frac{\partial p_2^B}{\partial a_1}}_{\substack{=0 \\ \text{first-order condition} \\ \text{at stage 2} \\ \text{(envelope theorem)}}} + \underbrace{\underbrace{\frac{\partial \Pi_2}{\partial p_1}}_{>0}\underbrace{\frac{\partial p_1^B}{\partial a_1}}_{<0}}_{\substack{<0 \\ \text{strategic effect} \\ \text{of positioning}}}.$$

Games in extensive form

overview

1. Examples: Non-simultaneous moves in simple bimatrix games
2. Example: the Stackelberg model
3. Transforming an extensive-form game into a strategic-form game
4. Subgame perfection and backward induction
5. Multi-stage games
6. Product differentiation
7. **Strategic trade policy**

Trade theory and policy
Introducing models of imperfect competition

- Thirty years ago, trade theory and policy were analyzed with models of perfect competition.
- Free trade was a usual implication of these models.
- Since the beginning of the 1980s, models and recommendations have changed. At first, the researchers used Cournot models.
    - Brander (1981) and Brander/Krugman (1983) show that free trade can lead to the exchange of identical products.
    - In another strand of the literature, Brander/Spencer (1981, 1983) reason that export subsidies can benefit exporting firms over and above the subsidies.
      —> strategic trade policy

- Two firms d and f in two countries d (Germany) and f (France) produce for a market in a third country (Italy).
- Inverse demand function $p(X) = a - bX$
- Identical marginal and average cost $c := c_d = c_f$ with $c < a$.
- The German government tries to maximize welfare by choosing an appropriate unit subsidy s benefitting its firm d. Welfare is given by

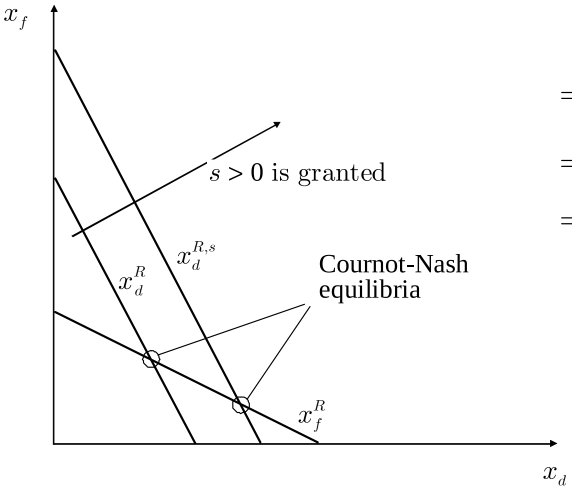
$$W(s) = \Pi_d^C(c - s, c) - s x_d^C(c - s, c).$$

- Optimal subsidy

$$s^* := \arg\max_{s \in \mathbb{R}} W(s) = \frac{a - c}{4} > 0.$$

- The subsidy has a direct effect on welfare and a strategic effect.
- The direct effect (holding the outputs constant): From our welfare point of view, it does not matter whether a sum of money ends up in the pockets of the domestic firm or in those of the government.
- The indirect effect (working through the firms' quantities): The subsidy amounts to a cost decrease for the domestic firm d :

$$\underbrace{\underbrace{\frac{\partial \Pi_d}{\partial x_f}}_{<0} \underbrace{\frac{\partial x_f^C}{\partial s}}_{<0}}_{>0} > 0,$$



$$
\begin{aligned}
& x_d^C\left(c - s^*, c\right) \\
= \; & \frac{a - c + 2s^*}{3b} \\
= \; & \frac{a - c}{2b} \\
= \; & x_d^S\left(c, c\right).
\end{aligned}
$$

- Eaton/Grossman (1986): Using price competition rather than quantity competition, the government should tax exports rather than handing out subsidies (see further exercises).
- Helpman/Krugman (1989): "One can always do better than free trade, but the optimal tariffs or subsidies seem to be small, the potential gains tiny, and there is plenty of room for policy errors that may lead to eventual losses rather than gains. [...] The case for free trade has always rested on an argument that it represents a good rule of thumb given uncertainty about the alternatives, realistic appreciation of the difficulties of managing political intervention, and the need to avoid trade wars."

Further exercises I

Problem 1

Analyze the sequential “head or tail” game where player 1 moves first.

Problem 2

Work through the innovation chapter in Pfähler/Wiese: Unternehmensstrategien im Wettbewerb.

Problem 3

Reconsider the police game where $0 < C < 4$ and $F > 1$ holds. Let the police be the first mover and assume that the indifferent agent abstains from committing a crime. Find the optimal control probability!

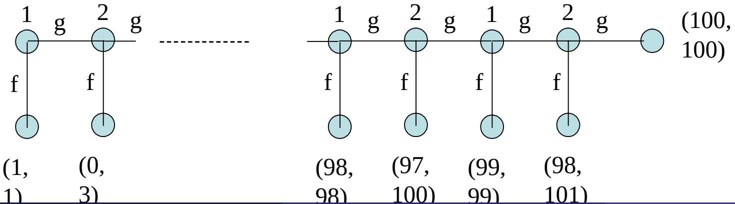
		agent	
		fraud	no fraud
police	control	$4 - C, 1 - F$	$4 - C, 0$
	no control	$0, 1$	$4, 0$

Further exercises II

Problem 4

Consider the centipede game! For every player, a strategy is a 99-tuple. For example, $\lfloor$g, g, g, g, f, ..., f$\rfloor$ is the strategy according to which a player chooses "go on" at his first four decision nodes and chooses "finish" at all the others.

- Which strategy would you choose if you were player 1?
- Can player 1's strategy $\lfloor$g, g, g, g, f, ..., f$\rfloor$ be part of a subgame-perfect strategy combination?
- Solve the centipede game by backward induction!
- Do you want to reconsider your answer to the first question?



Further exercises III

Problem 5

Two firms A and B

$p(Q) = 48 - Q$

$c = 12$

a) Reaction functions?
   Cournot outputs?

b) A Stackelberg leader
   Stackelberg outputs?
   Stackelberg equilibrium?

c) Cartel solution?

d) Perfect-competition quantity ($p = MC$)? Why are
   - Cournot results also called two-thirds solution and
   - Stackelberg results also called three-fourths solution?

Problem 6

Analyze the optimal subsidy for price competition on the Hotelling linear space

The model:

- Two firms d and f offer maximally differentiated products, $\Delta a = 1$. Therefore demand curves

$$\begin{aligned} x_d &= \frac{1}{2} + \frac{p_f - p_d}{2t} \text{ and} \\ x_f &= \frac{1}{2} + \frac{p_d - p_f}{2t}. \end{aligned}$$

- Units profits are $p_d - (c - s)$ for firm d and $p_f - c$ for firm f.
- Show: Equilibrium prices are

$$\begin{aligned} p_d^B &= t + c - \frac{2}{3}s \text{ and} \\ p_f^B &= t + c - \frac{1}{3}s. \end{aligned}$$

Problem 6 (sequel)

- Show: The resulting quantity supplied by firm d is

$$x_d^B = \frac{1}{2} + \frac{p_f^B - p_d^B}{2t} = \frac{1}{2} + \frac{1}{6}\frac{s}{t}.$$

- Trust $W(s) = \Pi_d^B(c - s, c) - s x_d^B(c - s, c) = \left[t - \frac{2}{3}s\right]\left(\frac{1}{2} + \frac{1}{6}\frac{s}{t}\right)$
- Show: The welfare maximizing "subsidy" is

$$s^* = -\frac{3}{4}t.$$