Advanced Microeconomics

Games in strategic form

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Part C. Games and industrial organization

1. Games in strategic form
2. Price and quantity competition
3. Games in extensive form
4. Repeated games

Games in strategic form

overview

1. **Introduction, examples and definition**
2. Best responses (marking technique)
3. Dominance
4. Rationalizability
5. Nash equilibrium
6. Mixed-strategy Nash equilibria
7. Existence and number of mixed-strategy equilibria
8. Procedural rationality
9. Depictions
10. Critical reflections on game theory

Nobel prices in Game theory

1994

In 1994

'for their pioneering analysis of equilibria in the theory of non-cooperative games'

1/3 John C. Harsanyi (University of California, Berkeley),

1/3 John F. Nash (Princeton University), and

1/3 Reinhard Selten (Rheinische Friedrich-Wilhelms-Universität, Bonn).

In 2005

'for having enhanced our understanding of conflict and cooperation through game-theory analysis'

1/2 Robert J. Aumann (Hebrew University of Jerusalem), and

1/2 Thomas C. Schelling (University of Maryland, USA).

Some simple bimatrix games

stag hunt

		hunter 2	
		stag	hare
hunter 1	stag	5, 5	0, 4
	hare	4, 0	4, 4

Cooperation may pay, but may also fail.

Some simple bimatrix games

matching pennies/head or tail

		player 2	
		head	tail
player 1	head	$1, -1$	$-1, 1$
	tail	$-1, 1$	$1, -1$

Police versus thief

- head = break-in or control at location "head"
- tail = break-in or control at location "tail"

Some simple bimatrix games

battle of the sexes

		he	
		theatre	football
she	theatre	4, 3	2, 2
	football	1, 1	3, 4

- Different standards
- Harmonizing laws in Europe

Some simple bimatrix games

game of chicken

		driver 2	
		continue	swerve
driver 1	continue	0, 0	4, 2
	swerve	2, 4	3, 3

- 1 and 2 approach a crossing (a parking spot). One speeds on and “wins”.
- 1 and 2 contemplate to open a pharmacy in a small town. The market is too small for both.

Some simple bimatrix games

prisoners'dilemma

		player 2	
		deny	confess
player 1	deny	4, 4	0, 5
	confess	5, 0	1, 1

Other examples:

1. Picking up one's waste (Pigovian tax, environmental laws)
2. Stealing cars (criminal law)
3. Paying taxes (tax law)

Definition of a game in strategic form

definition

Definition

A game in strategic form is a triple

$$\Gamma = \left(N, (S_i)_{i \in N}, (u_i)_{i \in N}\right) = (N, S, u).$$

- $N = \{1, ..., n\}$ – player set (nonempty and finite; $n := |N|$)
- S_i – i's strategy set
- $u_i : S \to \mathbb{R}$ – i's payoff function
- $S = \bigtimes_{i \in N} S_i$ – Cartesian product of all players' strategy sets with elements $s = (s_1, s_2, ..., s_n) \in S$
- Elements of S_i are called 'strategies'
- Elements of S are called 'strategy combinations'

Definition of a game in strategic form

example: battle of sexes

		he	
		theatre	football
she	theatre	4, 3	2, 2
	football	1, 1	3, 4

$N = \{she, he\}$, $S_{she} = S_{he} = \{\text{theatre,football}\}$, and u defined by

$$\begin{aligned} u_{she}\left(\text{theatre, theatre}\right) &= 4, u_{he}\left(\text{theatre, theatre}\right) = 3, \\ u_{she}\left(\text{theatre, football}\right) &= 2, u_{he}\left(\text{theatre, football}\right) = 2, \\ u_{she}\left(\text{football, theatre}\right) &= 1, u_{he}\left(\text{football, theatre}\right) = 1, \\ u_{she}\left(\text{football, football}\right) &= 3, u_{he}\left(\text{football, football}\right) = 4. \end{aligned}$$

Definition of a game in strategic form

notation

Removing player i's strategy from $s \in S$

- strategy combination s_{-i} of the remaining players from $N \backslash \{i\}$:

$$s_{-i} \in S_{-i} := \bigtimes_{\substack{j \in N, \\ j \neq i}} S_j$$

- player i's payoff: $u_i(s) = u_i(s_i, s_{-i})$

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3. Dominance
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Best responses

marking technique I

hunter 2

hunter 1

	stag	hare
stag	5, 5	0, 4
hare	4, 0	4, 4

	stag	hare
stag	5, 5 $\boxed{1}$	0, 4
hare	4, 0	4, 4 $\boxed{1}$

	stag	hare
stag	5, 5 $\boxed{1}\boxed{2}$	0, 4
hare	4, 0	4, 4 $\boxed{1}\boxed{2}$

Best responses

marking technique II

Problem

		player 2	
		left	*right*
player 1	*up*	$1, -1$	$-1, 1$
	down	$-1, 1$	$1, -1$

	player 2	
	left	*right*
up	$4, 4$	$0, 5$
down	$5, 0$	$1, 1$

Best responses

marking technique III

Solution

player 2

		left	*right*
pl. 1	*up*	$1,-1\ \boxed{\mathit{1}}$	$-1,1\ \boxed{\mathit{2}}$
	down	$-1,1\ \boxed{\mathit{2}}$	$1,-1\ \boxed{\mathit{1}}$

player 2

	left	*right*
u.	$4,4$	$0,5\ \boxed{\mathit{2}}$
d.	$5,0\ \boxed{\mathit{1}}$	$1,1\ \boxed{\mathit{1}}\boxed{\mathit{2}}$

- left: no dominant strategies, no Nash equilibrium
- right: s_2^2 dominant and $\left(s_1^2, s_2^2\right)$ Nash equilibrium

Best responses

marking technique IV

Definition

The function $s_i^R : S_{-i} \to 2^{S_i}$ given by

$$s_i^R\left(s_{-i}\right) := \arg\max_{s_i \in S_i} u_i\left(s_i, s_{-i}\right)$$

– best-response function (a best response, a best answer) for player $i \in N$.

$=$ marking technique

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Dominance

definition

Let (N, S, u) be a game, $i \in N$.

Definition

Strategy $s_i \in S_i$ (weakly) dominates $s_i' \in S_i$ if

(i) $u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i})$ for all $s_{-i} \in S_{-i}$ and

(ii) $u_i(s_i, s_{-i}) > u_i(s_i', s_{-i})$ for at least one $s_{-i} \in S_{-i}$.

Definition

Strategy $s_i \in S_i$ strictly dominates $s_i' \in S_i$ if
$u_i(s_i, s_{-i}) > u_i(s_i', s_{-i})$ for all $s_{-i} \in S_{-i}$.

- s_i' is (weakly/strictly) dominated (by s_i)
- If s_i (weakly/strictly) dominates all $s_i' \in S_i$, s_i is (weakly/strictly) dominant.

Dominance

the prisoners' dilemma

		firm 2	
		high	low
firm 1	high	$4, 4$	$0, 5$
	low	$5, 0$	$1, 1$

- Individual rationality vs. collective rationality
- How to achieve payoff tuple $(4, 4)$?
  - twin argument: both apply same reasoning
    $\Rightarrow$ non-diagonal outcomes can be disregarded
  - promise to cooperate
  - reputation
  - repeated games
  - altruism

Dominance

exercise

Problem

Is the stag hunt solvable by dominance arguments? How about head or tail, game of chicken, or the battle of the sexes?

	stag	*hare*
stag	5, 5	0, 4
hare	4, 0	4, 4

	head	*tail*
head	1, −1	−1, 1
tail	−1, 1	1, −1

	continue	*swerve*
continue	0, 0	4, 2
swerve	2, 4	3, 3

	theatre	*football*
theatre	4, 3	2, 2
football	1, 1	3, 4

The second-price auction

the game and a claim

- bidders $i = 1, 2$
- r_i – i's reservation price (= willingness to pay)
- $S_i = [0, +\infty)$ – i hands in a (sealed) bid
- $s_i < s_j$ makes j get the object at price s_i

$$u_1(s_1, s_2) = \begin{cases} 0, & s_1 < s_2, \\ \frac{1}{2}(r_1 - s_2), & s_1 = s_2, \\ r_1 - s_2, & s_1 > s_2 \end{cases}$$

Claim: $s_1 := r_1$ is a dominant strategy.

The second-price auction

proof of the claim

1. $r_1 < s_2$
   $s_1 = r_1$ —> payoff 0
   $s_1 > r_1$ and $s_1 < s_2$ —> payoff 0
   $s_1 > r_1$ and $s_1 \geq s_2$ —> payoff < 0
   $s_1 < r_1$ —> payoff 0

2. $r_1 = s_2$
   Expected payoff is 0, no matter how s_1 is choosen. Do you see, why?

Problem

Show that $s_1 = r_1$ *is a dominant strategy in case of* $r_1 > s_2$.

$\Rightarrow$ The auction game due to Vickrey is dominance solvable.

Take it or leave it

		player 2 accepts if he is offered at least ... coins				player 2 does not accept
		0	1	2	3	
player 1	0	$(3,0)$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$
offers	1	$(2,1)$	$(2,1)$	$(0,0)$	$(0,0)$	$(0,0)$
player 2	2	$(1,2)$	$(1,2)$	$(1,2)$	$(0,0)$	$(0,0)$
... coins	3	$(0,3)$	$(0,3)$	$(0,3)$	$(0,3)$	$(0,0)$

Take it or leave it

iterated dominance I

Deleting the last three columns yields

		player 2 accepts if he is offered at least … coins	
		0	1
player 1	0	(3, 0)	(0, 0)
offers	1	(2, 1)	(2, 1)
player 2	2	(1, 2)	(1, 2)
… coins	3	(0, 3)	(0, 3)

Take it or leave it

iterated dominance II

Deleting the last two rows

		player 2 accepts if he is offered at least ... coins	
		0	1
player 1	0	$(3, 0)$	$(0, 0)$
offers	1	$(2, 1)$	$(2, 1)$

Procedure stops.

The Basu game

the story

- $i \in \{1, 2\}$
- $S_i = \{2, 3, ..., 100\}$
- both get the lowest figure adjusted by an honesty premium/dishonesty punishment of 2

$$u_1(s_1, s_2) = \begin{cases} s_1 + 2, & \text{if} \quad s_1 < s_2, \\ s_1, & \text{if} \quad s_1 = s_2, \\ s_2 - 2, & \text{if} \quad s_1 > s_2; \end{cases}$$

The Basu game

the matrix

Trav.1	Traveler 2 requests so many coins						
claims	2	3	4	$\cdots$	98	99	100
2	$(2,2)$	$(4,0)$	$(4,0)$	$(4,0)$	$(4,0)$	$(4,0)$	$(4,0)$
3	$(0,4)$	$(3,3)$	$(5,1)$	$(5,1)$	$(5,1)$	$(5,1)$	$(5,1)$
4	$(0,4)$	$(1,5)$	$(4,4)$	$(6,2)$	$(6,2)$	$(6,2)$	$(6,2)$
$\vdots$	$(0,4)$	$(1,5)$	$(2,6)$				
98	$(0,4)$	$(1,5)$	$(2,6)$		$(98,98)$	$(100,96)$	$(100,96)$
99	$(0,4)$	$(1,5)$	$(2,6)$		$(96,100)$	$(99,99)$	$(101,97)$
100	$(0,4)$	$(1,5)$	$(2,6)$		$(96,100)$	$(97,101)$	$(100,100)$

Problem

Any dominated strategies?

The Basu game

somewhat reduced

Trav.1	Traveler 2 requests so many coins						
claims	2	3	4	$\cdots$	97	98	99
2	$(2,2)$	$(4,0)$	$(4,0)$	$(4,0)$	$(4,0)$	$(4,0)$	$(4,0)$
3	$(0,4)$	$(3,3)$	$(5,1)$	$(5,1)$	$(5,1)$	$(5,1)$	$(5,1)$
4	$(0,4)$	$(1,5)$	$(4,4)$	$(6,2)$	$(6,2)$	$(6,2)$	$(6,2)$
$\vdots$	$(0,4)$	$(1,5)$	$(2,6)$				
97	$(0,4)$	$(1,5)$	$(2,6)$		$(97,97)$	$(99,95)$	$(99,95)$
98	$(0,4)$	$(1,5)$	$(2,6)$		$(95,99)$	$(98,98)$	$(100,96)$
99	$(0,4)$	$(1,5)$	$(2,6)$		$(95,99)$	$(96,100)$	$(99,99)$

The Basu game

more reduced

		Traveler 2	
		2	3
Traveler 1	2	$(2, 2)$	$(4, 0)$
	3	$(0, 4)$	$(3, 3)$

Problem

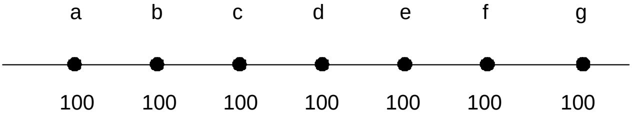
Do you know this game?

Other examples

- Clarke-Groves mechanism
- Cournot-Dyopol

Rationalizability

Two pub owners need to choose a location.



Apply iterated rationalizability!

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Nash equilibrium

	continue	swerve
continue	0, 0	4, 2 $\boxed{1}$ $\boxed{2}$
swerve	2, 4 $\boxed{1}$ $\boxed{2}$	3, 3

- Reaction function for driver 1

“swerve if driver 2 continues;
continue if driver 2 swerves”

- Similarly for driver 2

Nash equilibrium:
“Intersection” of the two reaction functions, i.e., strategy combinations

- (swerve, continue) and
- (continue, swerve)

Nash equilibrium

Definition

$s^* = (s_1^*, s_2^*, ..., s_n^*) \in S$ is a Nash equilibrium if for all i from N

$$u_i\left(s_i^*, s_{-i}^*\right) \geq u_i\left(s_i, s_{-i}^*\right)$$

holds for all s_i from S_i.

Nash equilibrium =

- mutually best responses
- nobody has an incentive to deviate

Problem

Use best-response functions to characterize $(s_1^*, s_2^*, ..., s_n^*)$ *as a Nash equilibrium!*

Nash equilibrium

exercise

Problem

Find all the Nash equilibria!

	stag	*hare*
stag	5, 5	0, 4
hare	4, 0	4, 4

	head	*tail*
head	1, −1	−1, 1
tail	−1, 1	1, −1

	continue	*swerve*
continue	0, 0	4, 2
swerve	2, 4	3, 3

	theatre	*football*
theatre	4, 3	2, 2
football	1, 1	3, 4

Nash equilibrium

equilibrium in dominant strategies

		player 2	
		deny	confess
player 1	deny	4, 4	0, 5
	confess	5, 0	1, 1

How about

	left	right
up	4, 4	4, 4
down	0, 0	4, 4

Nash equilibrium

equilibrium in the basu game

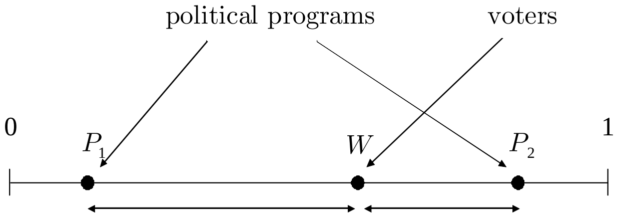
Problem

Find the equilibrium!

	2	*3*	*4*	…	*98*	*99*	*100*
2	(2, 2)	(4, 0)	(4, 0)	(4, 0)	(4, 0)	(4, 0)	(4, 0)
3	(0, 4)	(3, 3)	(5, 1)	(5, 1)	(5, 1)	(5, 1)	(5, 1)
4	(0, 4)	(1, 5)	(4, 4)	(6, 2)	(6, 2)	(6, 2)	(6, 2)
⋮	(0, 4)	(1, 5)	(2, 6)				
98	(0, 4)	(1, 5)	(2, 6)		(98, 98)	(100, 96)	(100, 96)
99	(0, 4)	(1, 5)	(2, 6)		(96, 100)	(99, 99)	(101, 97)
100	(0, 4)	(1, 5)	(2, 6)		(96, 100)	(97, 101)	(100, 100)

Political parties

two parties



- One-dimensional political space (left - right)
- Voters prefer the program closest to their political preferences.
- Even distribution between 0 (extreme left) and 1 (extreme right).
- Parties choose programs P_1 and P_2, respectively.

Two political parties

median voter

Theorem

In the above model, there exists exactly one equilibrium: both parties choose the middle position $\frac{1}{2}$.

Proof.

- In equilibrium, we have $P_1 = P_2$. Otherwise ...
- In equilibrium, we have $P_1 = P_2 = \frac{1}{2}$. Otherwise ...
- There is at most one equilibrium.
- $(P_1, P_2) = \left(\frac{1}{2}, \frac{1}{2}\right)$ is an equilibrium.
  - If party 1 deviates, ...
  - If party 2 deviates, ...

☐

Three political parties

Theorem

There is no equilibrium with three political parties.

Proof.

There is no equilibrium at

- $P_1 \neq P_2 \neq P_3$
- $P_1 = P_2 \neq P_3$
- $P_1 = P_2 = P_3$
  - $= \frac{1}{2}$
  - $\neq \frac{1}{2}$

□

Instability of political programs

is a theoretical phenomenon with practical relevance:

- internal party strife
- median-voter orientation
- new parties at the left or right edge

But: Political parties cannot change their programs arbitrarily.

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Mixed strategies

introductory remarks

- Randomization as an observable phenomenon
- Interpretations
    - Choosing probability distributions
    - Player i's choice depends on information unknown to player j

Mixed strategies

definitions

Definition

Let σ_i be a probability distribution on S_i:

$$\sigma_i\left(s_i^j\right) \geq 0 \text{ for all } j = 1, ..., |S_i| \text{ and } \sum_{j=1}^{|S_i|} \sigma_i\left(s_i^j\right) = 1$$

– σ_i is a mixed strategy.

Notation:

Σ_i – set of i's mixed strategies

$\Sigma := \bigtimes_{i \in N} \Sigma_i$ – set of combinations of mixed strategies

$\Sigma_{-i} := \bigtimes_{j \in N, j \neq i} \Sigma_j$ – set of combinations of mixed strategies of players other than i

Mixed strategies

remarks

- we may write

$$\sigma_i = \left(\sigma_i\left(s_i^1\right), \sigma_i\left(s_i^2\right), ..., \sigma_i\left(s_i^{|S_i|}\right)\right)$$

- call σ_i properly mixed if $\sigma_i(s_i) \neq 1$ for all s_i
- identify s_i^1 with $(1, 0, 0, ..., 0)$
- Σ contains n-tuples $\sigma = (\sigma_1, \sigma_2, ..., \sigma_n)$
- Σ_{-i} is set of $n - 1$-tuples $\sigma_{-i} = (\sigma_1, \sigma_2, ..., \sigma_{i-1}, \sigma_{i+1}, ..., \sigma_n)$

Mixed strategies

expected payoffs: example

Problem

Calculate the expected payoff for player 1 if player 1 chooses theatre with probability $\frac{1}{2}$ and player 2 chooses theatre with probability $\frac{1}{3}$!

	theatre	*football*
theatre	$4,3$	$2,2$
football	$1,1$	$3,4$

Mixed strategies

expected payoffs

Lemma

The payoff for a mixed strategy is the mean of the payoffs for the pure strategies:

$$u_i\left(\sigma_i,\sigma_{-i}\right)=\sum_{j=1}^{|S_i|}\sigma_i\left(s_i^j\right)u_i\left(s_i^j,\sigma_{-i}\right)$$

Mixed strategies

nash equilibrium

Definition

The strategy combination

$$\sigma^* = (\sigma_1^*, \sigma_2^*, ..., \sigma_n^*) \in \Sigma$$

is a Nash equilibrium in mixed strategies if for all i from N

$$u_i\left(\sigma_i^*, \sigma_{-i}^*\right) \geq u_i\left(\sigma_i, \sigma_{-i}^*\right)$$

holds for all σ_i from Σ_i.

Or: $\sigma_i^* \in \sigma_i^R\left(\sigma_{-i}^*\right)$ for all $i \in N$.

Equilibria

finding equilibria in mixed strategies

Idea

- Fix $\bar{\sigma}_{-i}$.
- Then:

$$u_i\left(\sigma_i,\bar{\sigma}_{-i}\right)=\sum_{j=1}^{|S_i|}\sigma_i\left(s_i^j\right)u_i\left(s_i^j,\bar{\sigma}_{-i}\right).$$

- If $u_i\left(s_i^k,\bar{\sigma}_{-i}\right)>u_i\left(s_i^l,\bar{\sigma}_{-i}\right)$ and $\sigma_i\left(s_i^l\right)\neq 0$, σ_i is not a best answer to $\bar{\sigma}_{-i}$.
- 2×2-games: properly mixed Nash equilibria obey

$$u_i\left(s_i^1,\sigma_{-i}^*\right)=u_i\left(s_i^2,\sigma_{-i}^*\right).$$

Equilibria

finding equilibria in mixed strategies: 'head or tail' I

	head	tail
head	$1, -1$	$-1, 1$
tail	$-1, 1$	$1, -1$

$$\begin{aligned}
& u_1(\text{head}, \sigma_2) \geq u_1(\text{tail}, \sigma_2) \\
\Leftrightarrow\quad & \sigma_2 \cdot 1 + (1-\sigma_2)\cdot(-1) \geq \sigma_2 \cdot (-1) + (1-\sigma_2)\cdot 1 \\
\Leftrightarrow\quad & 2\sigma_2 - 1 \geq -2\sigma_2 + 1 \\
\Leftrightarrow\quad & \sigma_2 \geq \frac{1}{2}
\end{aligned}$$

$$\sigma_1^R(\sigma_2) = \begin{cases} 1, & \sigma_2 > \frac{1}{2} \\ [0,1], & \sigma_2 = \frac{1}{2} \\ 0, & \sigma_2 < \frac{1}{2} \end{cases}$$

Equilibria

finding equilibria in mixed strategies: ‘head or tail’ II

Similarly,

$$\sigma_2^R\left(\sigma_1\right)=\begin{cases}0, & \sigma_1>\frac{1}{2}\\ \left[0,1\right], & \sigma_1=\frac{1}{2}\\ 1, & \sigma_1<\frac{1}{0}\end{cases}$$

- Equilibrium $\left(\left(\frac{1}{2},\frac{1}{2}\right),\left(\frac{1}{2},\frac{1}{2}\right)\right)$ or $\left(\frac{1}{2},\frac{1}{2}\right)$ (different notations)
- No other equilibrium:
  - $\sigma_1>\frac{1}{2}\rightarrow\sigma_2=\sigma_2^R\left(\sigma_1\right)=0\rightarrow\sigma_1=\sigma_1^R\left(\sigma_2\right)=0$
  - $\sigma_1<\frac{1}{2}\rightarrow\sigma_2=\sigma_2^R\left(\sigma_1\right)=1\rightarrow\sigma_1=\sigma_1^R\left(\sigma_2\right)=1$

Equilibria

finding equilibria in mixed strategies: 'battle of the sexes'

	theatre	football
theatre	4, 3	2, 2
football	1, 1	3, 4

- $u_1(\sigma_1, \sigma_2) = 4\sigma_1\sigma_2 + 2\sigma_1(1-\sigma_2) + (1-\sigma_1)\sigma_2 + 3(1-\sigma_1)(1-\sigma_2)$

- $\frac{\partial u_1}{\partial \sigma_1} = 4\sigma_2 + 2(1-\sigma_2) - \sigma_2 - 3(1-\sigma_2) = 4\sigma_2 - 1 \begin{cases} <0, & \sigma_2 < \frac{1}{4} \\ =0, & \sigma_2 = \frac{1}{4} \\ >0, & \sigma_2 > \frac{1}{4} \end{cases}$

- $\sigma_1^R(\sigma_2) = \begin{cases} 0, & \sigma_2 < \frac{1}{4} \\ [0,1], & \sigma_2 = \frac{1}{4} \\ 1, & \sigma_2 > \frac{1}{4} \end{cases}$

Equilibria

finding equilibria in mixed strategies: exercise

Problem

Find all (mixed) equilibria! Draw the best responses!

	left	*right*
up	5, 5	0, 4
down	4, 0	4, 4

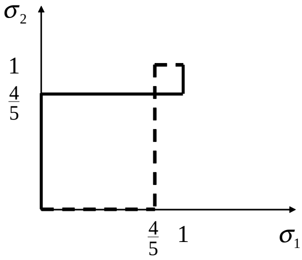
	left	*right*
up	1, 1	1, 1
down	1, 1	0, 0

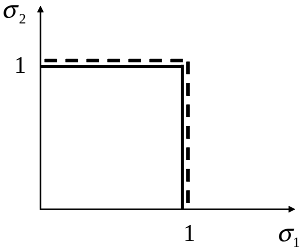
	left	*right*
up	4, 3	2, 2
down	1, 1	3, 4

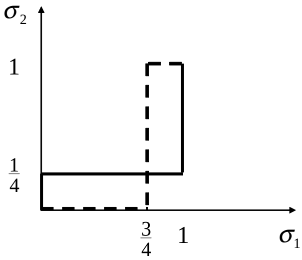
	left	*right*
up	4, 4	0, 5
down	5, 0	1, 1

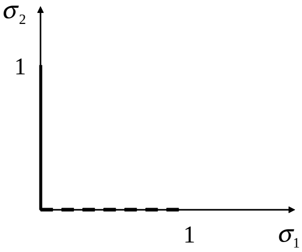
Equilibria

finding equilibria in mixed strategies: exercise









The police game

the model

		agent	
		fraud	no fraud
police	control	$4-C, 1-F$	$4-C, 0$
	no control	$0, 1$	$4, 0$

- C – control costs $(0 < C < 4)$
- F – punishment $(F > 1)$

Problem

Find all the pure-strategy equilibria!

The police game

equilibria in mixed strategies

	fraud	no fraud
control	$4-C, 1-F$	$4-C, 0$
no control	$0, 1$	$4, 0$

- $u_p\left(\text{control}, \sigma_a\right) \overset{!}{=} u_p\left(\text{no control}, \sigma_a\right)$
- $\sigma_a\left(4-C\right) + \left(1-\sigma_a\right)\left(4-C\right) \overset{!}{=} \sigma_a \cdot 0 + \left(1-\sigma_a\right)4 \quad \Leftrightarrow \quad \sigma_a \overset{!}{=} \frac{C}{4}$

Problem

Which controlling probability σ_p chosen by the police makes the agent indifferent between committing and not committing the crime?

The police game

equilibria in mixed strategies: payoffs

Solution

We calculate:

$$\sigma_p (1-F) + (1-\sigma_p) 1 \overset{!}{=} \sigma_p \cdot 0 + (1-\sigma_p) \cdot 0 \quad \Leftrightarrow \quad \sigma_p \overset{!}{=} \frac{1}{F}.$$

In equilibrium, the payoffs are

$$u_p = \frac{1}{F}(4-C) + \left(1-\frac{1}{F}\right)\frac{C}{4} \cdot 0 + \left(1-\frac{1}{F}\right)\left(1-\frac{C}{4}\right) 4 = 4-C$$

for the police and

$$u_a = \frac{C}{4}\frac{1}{F}(1-F) + \frac{C}{4}\left(1-\frac{1}{F}\right) 1 + \left(1-\frac{C}{4}\right) \cdot 0 = 0$$

for the prospective criminal. (Short-cut?)

The volunteer's dilemma

	volunteer	defect
volunteer	2, 2	2, 5
defect	5, 2	0, 0

Mixed-strategy equilibrium: Let σ be the probability of volunterring. Player 1 weakly prefers volunteering if

$$\sigma \cdot 2 + (1 - \sigma) \cdot 2 \geq \sigma \cdot 5 + (1 - \sigma) \cdot 0,$$

i.e., if

$$\sigma \leq \frac{2}{5}.$$

By symmetry, the only mixed-strategy equilibrium is

$$\left(\frac{2}{5}, \frac{2}{5} \right)$$

The volunteer's dilemma

	volunteer	defect
volunteer	$2-C_1, 2-C_2$	$2-C_1, 5$
defect	$5, 2-C_2$	$0, 0$

Mixed-strategy equilibrium: Let σ be the probability of volunterring. Player 1 weakly prefers volunteering if

$$2 - C_1 \geq \sigma \cdot 5 + (1-\sigma) \cdot 0,$$

i.e., if

$$\sigma \leq \frac{2-C_1}{5}.$$

The only mixed-strategy equilibrium is

$$\left(\frac{2}{5} - \frac{1}{5} C_2, \frac{2}{5} - \frac{1}{5} C_1 \right)$$

Thus: $C_1 < C_2$ implies that player 1 (with low cost of volunteering) volunteers with a smaller probability than player 2!

Games in strategic form

overview

1. Introduction, examples and definition
2. Best responses (marking technique)
3. Dominance
4. Rationalizability
5. Nash equilibrium
6. Mixed-strategy Nash equilibria
7. **Existence and number of mixed-strategy equilibria**
8. Procedural rationality
9. Depictions
10. Critical reflections on game theory

Existence and number of mixed-strategy equilibria

number I

Definition

$\Gamma = (N, S, u)$ and $|S| < \infty$ (or $|S_i| < \infty$ for all $i \in N$)
– finite game in strategic form.

- n payoffs for each strategy combination define a game
- $|S| = |S_1| \cdot |S_2| \cdot \ldots \cdot |S_n|$

- Hence: a point in $\mathbb{R}^{n \cdot |S|}$ represents a game

Existence and number of mixed-strategy equilibria

number II

Theorem

Nearly all finite strategic games have a finite and odd number of equilibria in mixed strategies.

- $\Gamma^* \in \mathbb{R}^{n \cdot |S|}$ with odd number of equilibria
  $\Rightarrow$ all Γ in some ε-ball around Γ^* have the same number of equilibria
- $\Gamma^* \in \mathbb{R}^{n \cdot |S|}$ with infinite or even number of equilibria
  $\Rightarrow$ there is Γ with odd number of equilibria in every ε-ball around Γ^*

Existence and number of mixed-strategy equilibria

number: example I

		player 2	
		left	right
pl. 1	up	$1, 1$	$0, 0$
	down	$0, 0$	$0, 0$

		player 2	
		left	right
pl. 1	up	$1, 1$	α, α
	down	$0, 0$	β, β

- How many equilibria in the left-hand game?
- $0 < \alpha < 1$ and $\beta \leq 0$: how many?
- $\alpha = 0$ and $\beta < 0$: how many?
- $\alpha = 0$ and $\beta > 0$: how many?

Existence and number of mixed-strategy equilibria

number: example II

		player 2	
		left	right
pl. 1	up	$1,1$	$1,1$
	down	$1,1$	$0,0$

		player 2	
		left	right
pl. 1	up	$1+\varepsilon, 1+\varepsilon$	$1,1$
	down	$1,1$	$0,0$

- How many equilibria in the left-hand game?
- $\varepsilon > 0$: how many?
- $\varepsilon < 0$: how many?

Existence and number of mixed-strategy equilibria

existence

Theorem (John Nash, 1950)

Any finite strategic game $\Gamma = (N, S, u)$ *has a mixed-strategy Nash equilibrium.*

Proof: see General equilibrium theory.

Economic genius: John Forbes Nash, Jr. I

- John Forbes Nash, Jr. (geb. 1928) is a US-american mathematician.
- After a very promising start of his career, he falls ill with schizophrenia and recovers in the 1990s.

Economic genius: John Forbes Nash, Jr. II

- Nash's personal history is the topic of the Hollywood film "A beautiful mind".
- Nash's dissertation at Princeton deals with game theory. Without knowing Cournot's work, he defines the equilibrium (later called: Nash equilibrium). He proves the above-mentioned theorem.

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Procedural rationality

introduction

Ariel Rubinstein an others have developed models of bounded rationality. In the context of mixed-strategies, Osborne and Rubinstein (AER) have developed the concept of procedural rationality. We present an example below that has been proposed by Tutic (Journal of Mathematical Sociology).

Osborne/Rubinstein:

... conditions of poor information ...

There is a large population of indivuals, pairs of whom are occasionally matched and interact. When entering the population, a player chooses her action after sampling each alternative once, picking the action that yields the highest payoff. An equilibrium corresponds to a steady state in which the probability that a new player chooses any given action is equal to the fraction of the population that currently chooses that action. ...

Procedural rationality

example: volunteer's dilemma

	volunteer	defect
volunteer	2, 2	2, 5
defect	5, 2	0, 0

Procedurally rational equilibrium: A new player tries volunteering and defecting once.

- Portion σ of the population volunteers $\Longrightarrow$ the new player prefers volunteering with probability $1 - \sigma$.
- Assume $\sigma = \frac{1}{3}$. Volunteering probability $>$ volunteering portion $\Rightarrow$ volunteering portion $\uparrow$.

Equilibrium defined by

$$\text{volunteering portion} \overset{!}{=} \text{probability of volunteering} \Leftrightarrow \sigma \overset{!}{=} 1 - \sigma$$

whence $\sigma = \frac{1}{2}$ in the procedurally rational equilibrium.

Procedural rationality

example: volunteer's dilemma with costs

	volunteer	defect
volunteer	$2-C_1, 2-C_2$	$2-C_1, 5$
defect	$5, 2-C_2$	$0, 0$

Assume $C_1 < 2$ and $C_2 < 2$. The players are still symmetric because we "interpret the payoff function as a representation of each player's ordial preferences. Then, the procedurally rational equilibrium is the same from above.

Thus: $C_1 < C_2$ implies that player 1 (with low cost of volunteering) volunteers with a the same probability as player 2!

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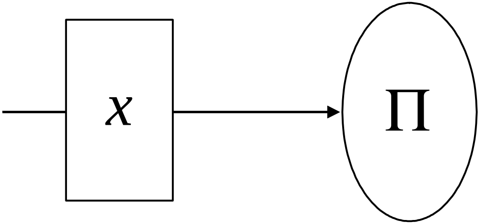
Depiction

decision theory

Trivial game with one player, only

Example monopoly:
x : quantity produced and sold
$\Pi(x)$: profit at x

Very simple depiction:



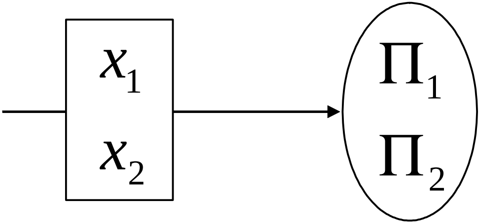
Depiction

game in strategic form

Simultaneous-move game with several players

Example duopoly:
x_1, x_2 : quantities produced and sold
$\Pi_1(x_1, x_2)$: firm 1's profit at (x_1, x_2)

Very simple depiction:



Depiction

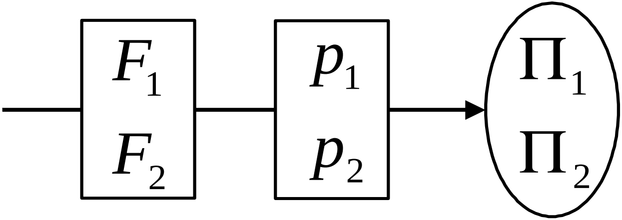
sequential-move game

Two-stage game with two firms that

- choose expenditures for R&D and then
- choose prices,

simultaneous at each stage

Very simple depiction:



F_1

F_2

p_1

p_2

Π_1

Π_2

—> chapter after next

Critical reflections on game theory

predictive power

Equilibria are meant to predict how players will act in specific situations. However, sometimes

- equilibria are counterintuitive (Basu game)
- we encounter multiple equilibria, as in these games:

	stag	hare
stag	5, 5	0, 4
hare	4, 0	4, 4

	continue	swerve
continue	0, 0	4, 2
swerve	2, 4	3, 3

	theatre	football
theatre	4, 3	2, 2
football	1, 1	3, 4

Further exercises I

Problem 1

Strategy combination (down, right) is a Nash equilibrium.

What can you say about the constants a, b, c and d?

		player 2	
		left	right
player 1	up	1, a	c, 1
	down	1, b	d, 1

Further exercises II

Problem 2
Consider a first price auction. There are $n = 2$ players $i = 1, 2$ who submit bids $b_i \geq 0$ simultaneously. Player i's willingness to pay for the object is given by w_i. Assume $w_1 > w_2 > 0$. The player with the highest bid obtains the object and has to pay his bid. If both players submit the highest bid, the object is given to player 1. The winning player $i \in \{1, 2\}$ obtains the payoff $w_i - b_i$ and the other the payoff zero. Determine the Nash equilibria in this game!

Problem 3

(a) Find a game in which player 1 has a weakly dominant strategy $\hat{s}_1$ (i.e., $\hat{s}_1$ weakly dominates all other strategies from S_1) and which exhibits two equilibria, one of which does not make use of $\hat{s}_1$.

(b) Is it possible that a player has a strictly dominant strategy that is not played in equilibrium?

Further exercises III

Problem 4

Read the opening scene of Mozart's and Schikaneder's "Magic Flute". Two players $i = 1, 2$ are involved in a dispute over an object. The willingness to pay for the object is w_i, $i = 1, 2$. Assume $w_1 \geq w_2 > 0$. Time is modeled as a continuous variable that starts at 0 and runs indefinitely. Each player i chooses the time $s_i \geq 0$ when to concede the object to the other player. Until the first concession each player loses one unit of payoff per unit of time. Player i's payoff function is given by

$$u_i\left(s_i, s_j\right) = \begin{cases} -s_i, & s_i < s_j \\ \frac{w_i}{2} - s_i, & s_i = s_j \\ w_i - s_j, & s_i > s_j. \end{cases}$$

Determine the Nash equilibria in this game!

Further exercises IV

Problem 5

Find all (mixed) Nash Equilibria of the following game:

		player 2		
		left	centre	right
player 1	up	$(4, 5)$	$(2, 1)$	$(4, 4)$
	middle	$(0, 1)$	$(1, 5)$	$(3, 2)$
	down	$(1, 1)$	$(0, 0)$	$(6, 0)$