

# Advanced Microeconomics

## Cost minimization and profit maximization

Harald Wiese

University of Leipzig

## Part B. Household theory and theory of the firm

1. The household optimum
2. Comparative statics and duality theory
3. Production theory
4. **Cost minimization and profit maximization**

# Cost minimization and profit maximization

## Overview

1. **Revisiting the production set**
2. Cost minimization
3. Long-run and short-run cost minimization
4. Profit maximization

# Definition of profit

## Definition

Let  $Z \subseteq \mathbb{R}^\ell$  be a production set and  $p \in \mathbb{R}_+^\ell$  a price vector  $\Rightarrow$

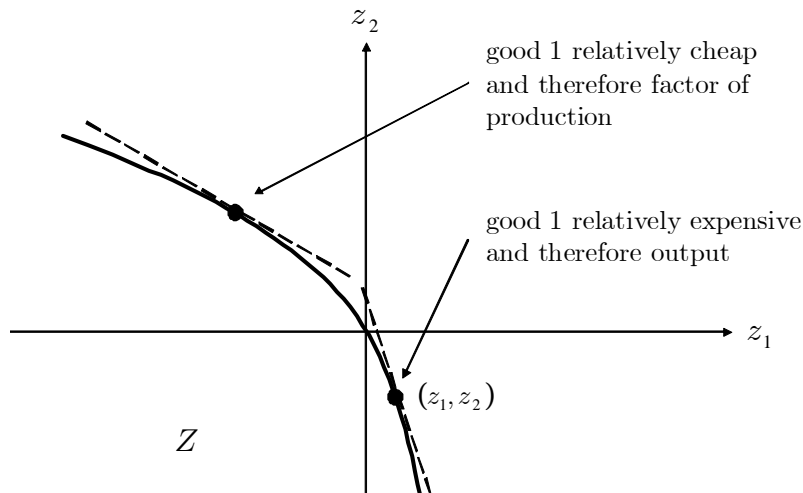
$$\Pi(z) := \underbrace{p \cdot z}_{\text{profit}} = \underbrace{\sum_{\substack{i=1, \\ z_i \geq 0}}^{\ell} p_i z_i}_{\text{revenue}} - \underbrace{\sum_{\substack{i=1, \\ z_i < 0}}^{\ell} p_i (-z_i)}_{\text{cost}}.$$

For a specific profit level  $\bar{\Pi}$ ,

$$\{z \in \mathbb{R}^\ell : p \cdot z = \bar{\Pi}\}$$

– the isoprofit line.

# Which good is an input and which an output?



# Revealed profit maximization

## Lemma

*Assume the existence of best elements in the production set  $Y$  for price changes of output and input goods that do not change the role of input and output goods  $\Rightarrow$*

- ▶ a price increase for a factor of production cannot increase demand for that factor and*
- ▶ a price increase for an output good cannot decrease the supply of that output good.*

# Revealed profit maximization

Proof.

Assume:

- ▶  $(p^A, w_1^A, w_2^A)$  and  $(p^B, w_1^B, w_2^B) \rightarrow$  price vectors;
- ▶  $(y^A, x_1^A, x_2^A)$  and  $(y^B, x_1^B, x_2^B) \rightarrow$  supply-and-demand vectors.

□

Then,  $(y^A, x_1^A, x_2^A)$  is best at  $(p^A, w_1^A, w_2^A)$  :

$$p^A y^A - w_1^A x_1^A - w_2^A x_2^A \geq p^A y^B - w_1^A x_1^B - w_2^A x_2^B.$$

Proof.

- ▶ Shuffling and reshuffling (see book) yields  $\Delta p \Delta y - \Delta w_1 \Delta x_1 - \Delta w_2 \Delta x_2 \geq 0$  where:
- ▶  $\Delta p := (p^A - p^B)$  ;
- ▶  $\Delta x_1 := x_1^A - x_1^B$ , etc.

□

# Cost minimization and profit maximization

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# Isocost line

## Definition

For a factor price vector  $w = (w_1, \dots, w_\ell) \in \mathbb{R}_+^\ell$ ,

$$w \cdot x$$

– the cost of using the factors of production  $x \in \mathbb{R}_+^\ell$ .

## Definition

For a specific level of cost  $\bar{C}$ ,

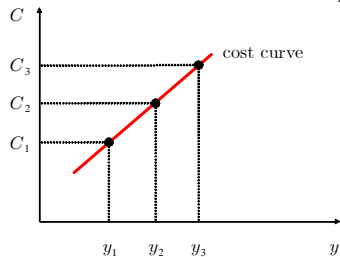
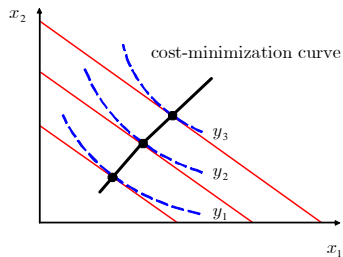
$$\left\{ x \in \mathbb{R}_+^\ell : w \cdot x = \bar{C} \right\}$$

– the isocost line.

## Problem

*Two factors of production 1 and 2. Slope of the isocost line? Hint: use the household analogy!*

# Isoclinic factor variation and the graphical derivation of the cost function



# Firm's cost-minimization problem and best-response function

## Definition

Assume

- ▶  $f$  – the production function  $\mathbb{R}_+^\ell \rightarrow \mathbb{R}_+$ ;
- ▶  $w \in \mathbb{R}_+^\ell$  – a vector of factor prices;
- ▶  $y \in \mathbb{R}_+$  – an element of  $f$ 's range, the output.

The firm's problem is to find the best-response function:

$$\chi^R(y) := \arg \min_{x \in \mathbb{R}_+^\ell} \{w \cdot x : f(x) \geq y\}.$$

Cost function

$$\begin{aligned} C &: \mathbb{R}_+ \rightarrow \mathbb{R}_+, \\ y &\mapsto C(y) = w \cdot \chi^R(y) \end{aligned}$$

## A comparison with household theory

	Household theory: expenditure min.	Theory of the firm: cost minimization
objective function	expenditure $p \cdot x$ (for the consumption of goods $x$ )	expenditure $w \cdot x$ (for the use of factors $x$ )
parameters	prices $p$ , utility $\bar{U}$ (indifference curve)	factor prices $w$ , output $y$ (isoquant)
first-order condition	$MRS \stackrel{!}{=} \frac{p_1}{p_2}$	$MRTS \stackrel{!}{=} \frac{w_1}{w_2}$
best bundle(s)	$\chi(p, \bar{U})$	$\chi^R(w, y)$ or $\chi^R(y)$
name of demand fct.	Hicksian demand	Hicksian factor demand
minimal value of objective function	$e(p, \bar{U})$ $= p \cdot \chi(p, \bar{U})$	$C(y) = C(w, y)$ $= w \cdot \chi^R(w, y)$

# Exercises

## Problem

*Fill in the missing term:*

$$\chi^R(y) = \left\{ x \in \mathbb{R}_+^\ell : f(x) \geq y \right. \\ \left. \text{and, for any } x' \in \mathbb{R}_+^\ell, f(x') \geq y \Rightarrow w \cdot x' \geq ?? \right\}$$

## Problem

*Define marginal cost and average cost.*

# A comparison with household theory

## Exercise

### Problem

Consider the production function  $f$  given by  $f(x_1, x_2) = x_1 + 2x_2$ .  
Find  $C(y)$ .

### Problem

$$MC = 2y$$

variable cost to produce 10 units?

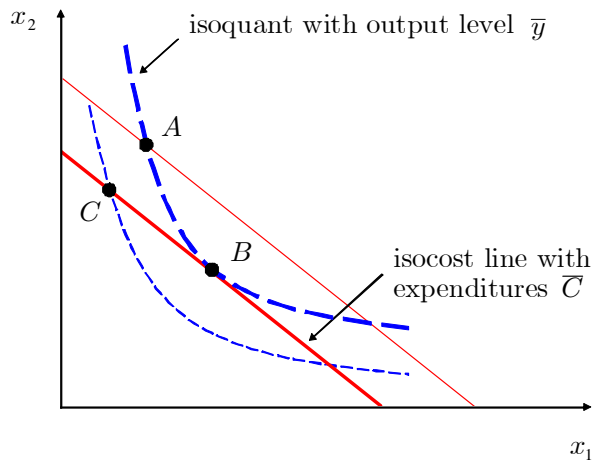
- ▶ continuous case (integral!) and
- ▶ discrete case (with cost of 2 for the first unit).

### Problem

$$y = f(x_1, x_2) = x_1^{\frac{1}{2}} + x_2^{\frac{1}{2}}$$

marginal-cost function?

# Cost-minimization and its dual I



## Cost-minimization and its dual II

	Household theory: utility maximization	Theory of the firm: output maximization
objective function	utility function	production function
parameters	prices $p$ , money budget $m$	factor prices $w$ , cost budget $\bar{C}$
first-order condition	$MRS \stackrel{!}{=} \frac{p_1}{p_2}$	$MRTS \stackrel{!}{=} \frac{w_1}{w_2}$
notation for best bundle(s)	$x(p, m)$	$x^R(w, \bar{C})$
name of demand function	Marshallian demand	Marshallian factor demand
maximal value of objective function	$V(p, m)$ $= U(x(p, m))$	$f(x^R(w, \bar{C}))$



# Cost minimization and profit maximization

## Overview

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3. **Long-run and short-run cost minimization**
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# Fixed factors and short-run cost function

## Definition

Assume two factors of production 1 and 2 at prices  $w_1$  and  $w_2$ .

- ▶ Factor 2 is fixed at  $\bar{x}_2 > 0$  if it cannot be reduced below  $\bar{x}_2$  “in the short run”.
- ▶ The short-run cost of using the factor combination  $(x_1, \bar{x}_2)$  is:

$$w_1 x_1 + w_2 \bar{x}_2.$$

- ▶ The short-run cost function is:

$$C_s(y, \bar{x}_2) := \min_{x_1 \in \mathbb{R}_+} \{w_1 x_1 + w_2 \bar{x}_2 : f(x) \geq y\}.$$

## Problem

$$f(x_1, x_2) = x_1^{\frac{1}{3}} x_2,$$

*short-run cost function  $C_s(y, \bar{x}_2)$ ?*

# Fixed and variable cost

## Definition

Let  $C_s : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  be a short-run cost function  $\Rightarrow$

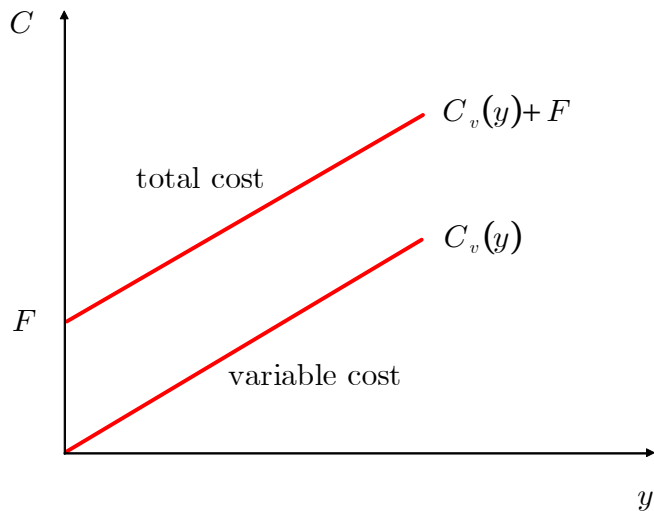
$$F := C_s(0)$$

– fixed cost

$$C_v(y) := C_s(y) - F$$

– the variable cost of producing  $y$ .

## Fixed and variable cost



# Quasi-fixed cost

## Definition

Let  $C : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  be a cost function that is not continuous at 0.

In case of  $C(0) = 0$  and  $\lim_{\substack{y \rightarrow 0, \\ y > 0}} C(y) > 0$ ,

$$F_q := \lim_{\substack{y \rightarrow 0, \\ y > 0}} C(y)$$

– quasi-fixed cost.

# Cost minimization and profit maximization

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# Profit maximization (output space)

Firm's profit in output space

## Definition

Let  $C : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  be a cost function  $\Rightarrow$

$$\begin{aligned} \Pi &: \mathbb{R}_+ \rightarrow \mathbb{R} \text{ and} \\ \underbrace{\Pi(y)}_{\text{profit}} &: = \underbrace{py}_{\text{revenue}} - \underbrace{C(y)}_{\text{cost}} \end{aligned}$$

– a firm's profit in output space.

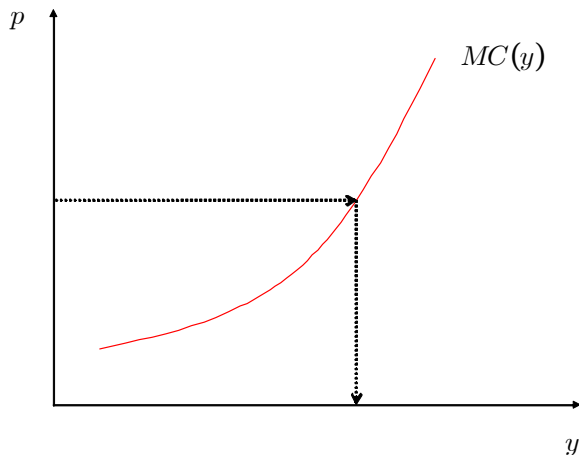
*FOC* for profit maximization:

$$MC \stackrel{!}{=} p.$$

# Profit maximization (output space)

## Firm's supply function I

In principle, the marginal-cost curve is the supply curve.





# Profit maximization (output space)

## Firm's supply function II

### Definition

Let  $C : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  be a (short-run or long-run) cost function  $\Rightarrow$

$$S : \mathbb{R}_+ \rightarrow \mathbb{R}_+, \\ p \mapsto S(p) := \arg \max_{y \in \mathbb{R}_+} \Pi(y)$$

– a firm's supply function.

But: if profit is negative at  $p = MC$ ?

# Profit maximization (output space)

## Firm's supply function III

- ▶ For the short-run supply:

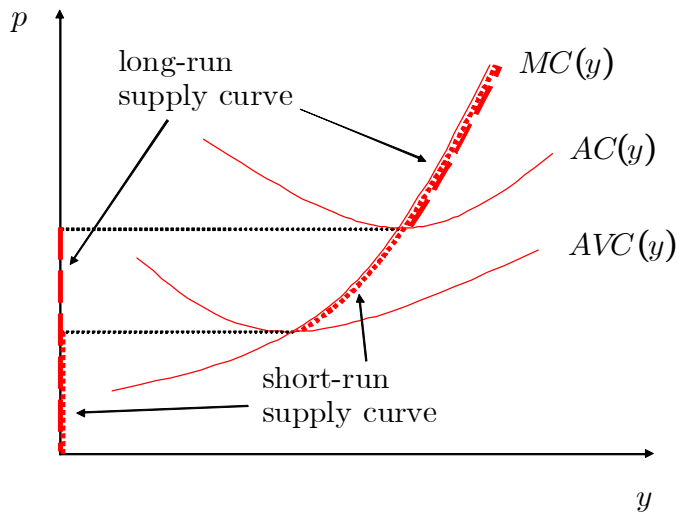
$$\begin{aligned} S_s(p) &= y^* \\ \Rightarrow \Pi_s(y^*) &\geq \Pi_s(0) = -F \\ \Leftrightarrow py^* - C_v(y^*) - F &\geq -F \\ \Leftrightarrow p &\geq \frac{C_v(y^*)}{y^*} =: AVC(y^*) \end{aligned}$$

$AVC(y)$  – average variable cost.

- ▶ For the long-run supply:

$$\begin{aligned} S(p) &= y^* \\ \Rightarrow \Pi(y^*) &\geq \Pi(0) = 0 \\ \Leftrightarrow py^* - C(y^*) &\geq 0 \\ \Leftrightarrow p &\geq \frac{C(y^*)}{y^*} = AC(y^*). \end{aligned}$$

## Profit maximization (output space)



# Profit maximization (output space)

## Exercise

### Problem

Consider the short-run cost function:

$$C_s(y) := 6y^2 + 15y + 54, y \geq 0$$

and the long-run cost function:

$$C(y) := \begin{cases} 6y^2 + 15y + 54, & y > 0 \\ 0, & y = 0 \end{cases}$$

Determine the short-run and the long-run supply functions  $S_s$  and  $S$ .

# Producer's rent

## Definition

The producer's rent at output price  $\hat{p}$  is:

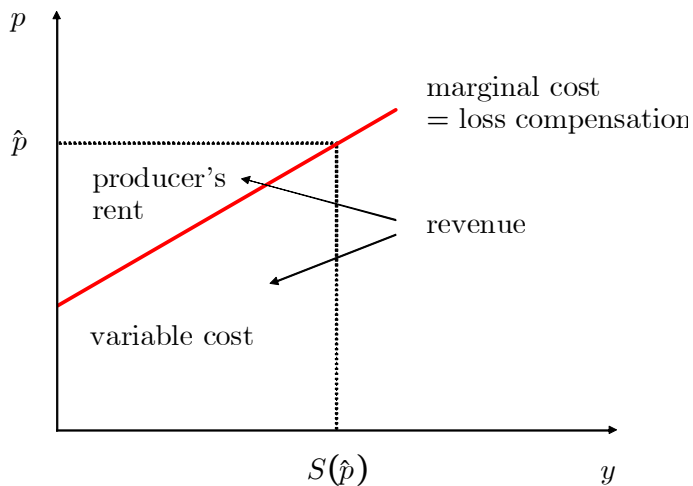
$$\begin{aligned} PR(\hat{p}) &: = CV(0 \rightarrow S(\hat{p})) \\ &= \hat{p}S(\hat{p}) - C_v(S(\hat{p})) \\ &= \int_0^{S(\hat{p})} [\hat{p} - MC(X)] dX. \end{aligned}$$

Consider

$$\begin{aligned} PR(\hat{p}) &= \hat{p}S(\hat{p}) - C_v(S(\hat{p})) \quad (\text{definition of producer's rent}) \\ &= \hat{p}S(\hat{p}) - [F + C_v(S(\hat{p}))] + F \quad (\text{adding } 0 = F - F) \\ &= \Pi(S(\hat{p})) + F \quad (\text{definition of profit}). \end{aligned}$$

- ▶ fixed cost: producer's rent equals profit plus fixed cost
- ▶ no fixed cost: producer's rent equals profit

## Producer's rent



# Profit maximization (input space)

Firm's profit in input space

## Definition

Let  $f : \mathbb{R}_+^\ell \rightarrow \mathbb{R}_+$  be a production function  $\Rightarrow$

$$\begin{aligned} \Pi &: \mathbb{R}_+^\ell \rightarrow \mathbb{R} \text{ and} \\ \underbrace{\Pi(x)}_{\text{profit}} &: = \underbrace{pf(x)}_{\text{revenue}} - \underbrace{w \cdot x}_{\text{cost}} \end{aligned}$$

– a firm's profit in input space.

*FOC* (partial differentiations):

$$MVP_i(x) := p MP_i = p \frac{\partial f}{\partial x_i} \stackrel{!}{=} w_i, i = 1, \dots, \ell$$

with  $MVP_i$  – factor  $i$ 's marginal value product at  $x$ .

# Profit maximization (input space)

Firm's factor demand function

## Definition

Let  $f : \mathbb{R}_+^\ell \rightarrow \mathbb{R}_+$  be a production function  $\Rightarrow$

$$D : \mathbb{R}_+^\ell \rightarrow \mathbb{R}_+^\ell,$$
$$w \mapsto D(w) := \arg \max_{x \in \mathbb{R}_+^\ell} \Pi(x)$$

– a firm's factor demand function.



# Profit maximization (input space)

## Exercises

### Problem

*Can you show that profit maximization implies cost minimization? (Hint: Divide the first-order conditions for profit maximization for two inputs!)*

### Problem

*A farmer has a cow that produces milk according to*

$$y_M = f(W, G) = W^{\frac{1}{4}} G^{\frac{1}{4}}$$

*where  $M$  stands for milk,  $W$  for water and  $G$  for grass.  
Determine the farmer's demand function for water.*

## Profit maximization with risk

All owners want their firm to maximize profit if

- ▶ prices are not affected by the firm's input and output choice,
- ▶ there is no risk involved and
- ▶ managers are fully controllable by owners.

# Profit maximization with risk

## Example

### Example

- ▶ agents *A* and *B* possess a firm;
- ▶ investment costs 100 ;
- ▶ returns 80 or 110, but probability distributions on returns differ.

	$w_1$ (80)	$w_2$ (110)	expected value of investment
<i>A</i> 's prob. distribution	$\frac{2}{10}$	$\frac{8}{10}$	$\frac{2}{10} \cdot 80 + \frac{8}{10} \cdot 110 - 100 = 4$
<i>B</i> 's prob. distribution	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2} \cdot 80 + \frac{1}{2} \cdot 110 - 100 = -5$

Similarly, different risk attitudes may lead to agents' holding opposing view on investment.

# Profit maximization with risk

## Arrow security

### Definition (Arrow security)

Let  $W = \{w_1, \dots, w_m\}$  be a set of  $m$  states of the world.

- ▶ The contingent good  $i \in \{1, \dots, m\}$  that pays one Euro in case of state of the world  $w_i$  and nothing in other states is called an Arrow security.
- ▶ If for each state of the world  $w_i$  an Arrow security  $i$  can be bought and sold for some given price  $p_i$ , we say that financial markets are complete.

### Problem

*Taking note of  $A'$  probability distribution, find his willingness to pay for the Arrow security 1. Hint: Calculate the expected gain from one unit of Arrow security 1.*

# Profit maximization with risk

## Arrow security

Assume  $m = 2$ . Arbitrage opportunity presented by

- ▶  $p_1 + p_2 < 1 \longrightarrow$  buy both Arrow securities!
- ▶  $p_1 + p_2 > 1 \longrightarrow$  sell both Arrow securities!

## Definition

Let  $W = \{w_1, \dots, w_m\}$  be a set of  $m$  states of the world with complete financial markets. The Arrow securities are said to be priced correctly (in equilibrium) if  $\sum_{i=1}^m p_i = 1$  holds.

- ▶ Thus, the prices of Arrow securities share the properties of probability distributions.
- ▶ Idea: The owners substitute the prices of the Arrow securities for their own probabilities. Then, their different beliefs are unimportant.

# Profit maximization with risk

Arrow security takes away risk

The owners  $A$  and  $B$  consider the investment package:

- ▶ spend 100 Euros on the investment,
- ▶ buy 20 units of Arrow good 1 and
- ▶ sell 10 units of Arrow good 2.

$$\text{State of the world } w_1 : \underbrace{80}_{\substack{\text{revenue} \\ \text{from} \\ \text{investment}}} + \underbrace{20}_{\substack{\text{revenue} \\ \text{from} \\ \text{Arrow good 1}}} = 100$$

$$\text{State of the world } w_2 : \underbrace{110}_{\substack{\text{revenue} \\ \text{from} \\ \text{investment}}} + \underbrace{-10}_{\substack{\text{revenue} \\ \text{from} \\ \text{Arrow good 2}}} = 100$$

Therefore, the two owners are indifferent between the two cases.  
The Arrow securities take all the risk away from them.

# Profit maximization with risk

## Arrow security

In case of  $w_1$ , the investment package is profitable iff

$$\begin{aligned} 0 &< \underbrace{80}_{\substack{\text{revenue} \\ \text{from} \\ \text{investment}}} - \underbrace{(100 + p_1 20 - p_2 10)}_{\substack{\text{cost of} \\ \text{investment} \\ \text{package}}} + \underbrace{20}_{\substack{\text{revenue} \\ \text{from} \\ \text{Arrow good 1}}} \\ &= -p_1 20 + p_2 10 \\ &= -p_1 20 + p_2 10 - 100 \underbrace{(1 - p_1 - p_2)}_0 \\ &= p_1 80 + p_2 110 - 100. \end{aligned}$$

holds, i.e., iff the investment's expected payoff (where Arrow prices take the role of probabilities) is positive.

# Profit maximization with risk

Arrow security

## Problem

*Show that the profitability of the investment package is equivalent to the investment criterion for state of the world 2, also.*



# The separation function of markets

Examples where markets allow a separation:

- ▶ If markets for consumption goods exist, a household with an endowment can consume his endowment but also any other bundle which does not cost more.
- ▶ If a market for manager effort is available, an owner-manager can buy additional (on top of his own) manager effort for his firm or supply effort to other firms.
- ▶ If Arrow securities are correctly priced, the profitability of an investment decision can be assessed separately from the beliefs and risk attitudes of the several owners.
- ▶ International trade allows an economy to consume a bundle different from the bundle produced.



## Further exercises I

### Problem 1

One firm with two factories

		1. unit	2. unit	3. unit	4. unit
marginal cost	factory 1	2	3	4	5
	factory 2	4	5	6	7

How to distribute 2 units among the two factories, how 4 units?

### Problem 2

$$y = f(x_1, x_2) = x_1^{\frac{1}{2}} x_2$$

short-run input  $x_2 = 50$

$$w_1 = 250, w_2 = 3$$

short-run marginal cost function SMC?

## Further exercises II

### Problem 3

$$y = f(x_1, x_2) = x_1^{\frac{1}{2}} x_2^{\frac{1}{2}}, w_1 = w_2 = 1$$

a) Assume  $\bar{x}_2 = 1$ . Sketch

- ▶ short-run variable average cost,
- ▶ short-run average cost,
- ▶ short-run marginal cost

and find the short-run supply curve!

b) Find the long-run supply curve!

## Further exercises III

### Problem 4

A firm has two factories  $A$  and  $B$  that obey the production functions  $f_A(x_1, x_2) = x_1 \cdot x_2$  and  $f_B(x_1, x_2) = x_1 + x_2$ , respectively. Given the factor-price ratio  $w = \frac{w_1}{w_2}$ , how to distribute output in order to minimize costs?

Hints:

- ▶ You are free to assume  $w_1 \leq w_2$ .
- ▶ Find the cost functions for the two factories.
- ▶ Are the marginal-cost curves upward-sloping or downward-sloping?