Advanced Microeconomics

Cost minimization and profit maximization

Harald Wiese

University of Leipzig

Part B. Household theory and theory of the firm

1. The household optimum
2. Comparative statics and duality theory
3. Production theory
4. **Cost minimization and profit maximization**

1. **Revisiting the production set**
2. Cost minimization
3. Long-run and short-run cost minimization
4. Profit maximization

Definition of profit

Definition

Let $Z \subseteq \mathbb{R}^\ell$ be a production set and $p \in \mathbb{R}^\ell_+$ a price vector $\Rightarrow$

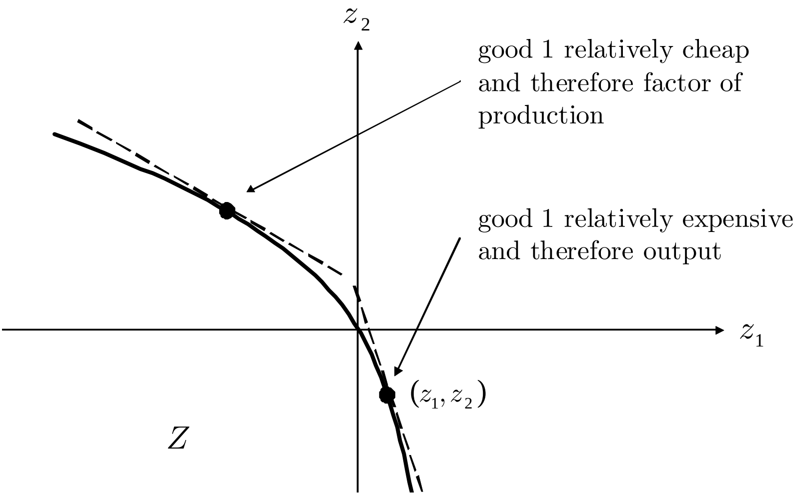
$$\Pi(z) := \underbrace{p \cdot z}_{\text{profit}} = \underbrace{\sum_{\substack{i=1,\\ z_i \geq 0}}^{\ell} p_i z_i}_{\text{revenue}} - \underbrace{\sum_{\substack{i=1,\\ z_i < 0}}^{\ell} p_i(-z_i)}_{\text{cost}}.$$

For a specific profit level $\bar{\Pi}$,

$$\left\{ z \in \mathbb{R}^\ell : p \cdot z = \bar{\Pi} \right\}$$

– the isoprofit line.

Which good is an input and which an output?



Revealed profit maximization

Lemma

Assume the existence of best elements in the production set Y for price changes of output and input goods that do not change the role of input and output goods $\Rightarrow$

- *a price increase for a factor of production cannot increase demand for that factor and*
- *a price increase for an output good cannot decrease the supply of that output good.*

Revealed profit maximization

Proof.

Assume:

- (p^A, w_1^A, w_2^A) and $(p^B, w_1^B, w_2^B) \longrightarrow$ price vectors;
- (y^A, x_1^A, x_2^A) and $(y^B, x_1^B, x_2^B) \longrightarrow$ supply-and-demand vectors.

□

Then, (y^A, x_1^A, x_2^A) is best at (p^A, w_1^A, w_2^A) :
$p^A y^A - w_1^A x_1^A - w_2^A x_2^A \geq p^A y^B - w_1^A x_1^B - w_2^A x_2^B$.

Proof.

- Shuffling and reshuffling (see book) yields $\Delta p \Delta y - \Delta w_1 \Delta x_1 - \Delta w_2 \Delta x_2 \geq 0$ where:
- $\Delta p := (p^A - p^B)$;
- $\Delta x_1 := x_1^A - x_1^B$, etc.

□

Cost minimization and profit maximization

Overview

1. Revisiting the production set
2. **Cost minimization**
3. Long-run and short-run cost minimization
4. Profit maximization

Isocost line

Definition

For a factor price vector $w = (w_1, ..., w_\ell) \in \mathbb{R}_+^\ell$,

$$w \cdot x$$

– the cost of using the factors of production $x \in \mathbb{R}_+^\ell$.

Definition

For a specific level of cost $\bar{C}$,

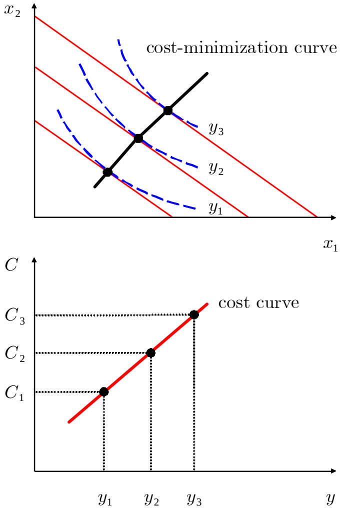
$$\left\{ x \in \mathbb{R}_+^\ell : w \cdot x = \bar{C} \right\}$$

– the isocost line.

Problem

Two factors of production 1 and 2. Slope of the isocost line? Hint: use the household analogy!

Isoclinic factor variation and the graphical derivation of the cost function



Firm's cost-minimization problem and best-response function

Definition

Assume

- f – the production function $\mathbb{R}_+^\ell \to \mathbb{R}_+$;
- $w \in \mathbb{R}_+^\ell$ – a vector of factor prices;
- $y \in \mathbb{R}_+$ – an element of f's range, the output.

The firm's problem is to find the best-response function:

$$\chi^R(y) := \arg\min_{x \in \mathbb{R}_+^\ell} \{w \cdot x \colon f(x) \geq y\}.$$

Cost function

$$\begin{aligned} C &: \mathbb{R}_+ \to \mathbb{R}_+, \\ & y \mapsto C(y) = w \cdot \chi^R(y) \end{aligned}$$

A comparison with household theory

	Household theory: expenditure min.	Theory of the firm: cost minimization
objective function	expenditure $p \cdot x$ (for the consumption of goods x)	expenditure $w \cdot x$ (for the use of factors x)
parameters	prices p, utility $\bar{U}$ (indifference curve)	factor prices w, output y (isoquant)
first-order condition	$MRS \overset{!}{=} \frac{p_1}{p_2}$	$MRTS \overset{!}{=} \frac{w_1}{w_2}$
best bundle(s)	$\chi\left(p, \bar{U}\right)$	$\chi^R\left(w, y\right)$ or $\chi^R\left(y\right)$
name of demand fct.	Hicksian demand	Hicksian factor demand
minimal value of objective function	$e\left(p, \bar{U}\right) = p \cdot \chi\left(p, \bar{U}\right)$	$C\left(y\right) = C\left(w, y\right) = w \cdot \chi^R\left(w, y\right)$

Exercises

Problem

Fill in the missing term:

$$\chi^R(y) = \Big\{x \in \mathbb{R}^\ell_+ \colon f(x) \geq y \\ \text{and, for any } x' \in \mathbb{R}^\ell_+,\ \ f(x') \geq y \Rightarrow w \cdot x' \geq ??\Big\}$$

Problem

Define marginal cost and average cost.

A comparison with household theory

Exercise

Problem

Consider the production function f given by $f(x_1, x_2) = x_1 + 2x_2$. Find $C(y)$.

Problem

$MC = 2y$

variable cost to produce 10 units?

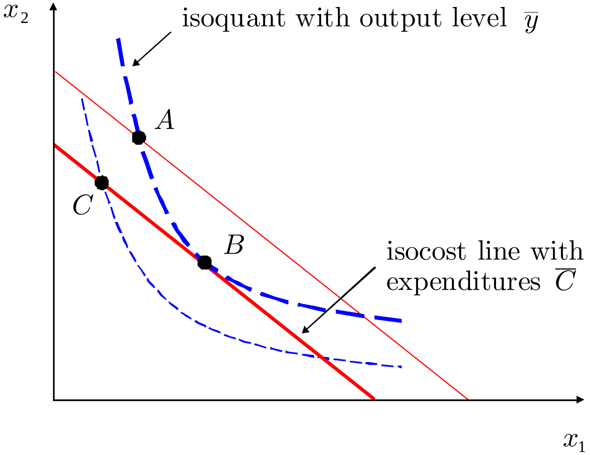
- *continuous case (integral!) and*
- *discrete case (with cost of 2 for the first unit).*

Problem

$y = f(x_1, x_2) = x_1^{\frac{1}{2}} + x_2^{\frac{1}{2}}$

marginal-cost function?

Cost-minimization and its dual I



Cost-minimization and its dual II

	Household theory: utility maximization	Theory of the firm: output maximization
objective function	utility function	production function
parameters	prices p, money budget m	factor prices w, cost budget $\bar{C}$
first-order condition	$MRS \overset{!}{=} \frac{p_1}{p_2}$	$MRTS \overset{!}{=} \frac{w_1}{w_2}$
notation for best bundle(s)	$x(p, m)$	$x^R(w, \bar{C})$
name of demand function	Marshallian demand	Marshallian factor demand
maximal value of objective function	$V(p, m) = U(x(p, m))$	$f(x^R(w, \bar{C}))$

1. Revisiting the production set
2. Cost minimization
3. **Long-run and short-run cost minimization**
4. Profit maximization

Fixed factors and short-run cost function

Definition

Assume two factors of production 1 and 2 at prices w_1 and w_2.

- Factor 2 is fixed at $\bar{x}_2 > 0$ if it cannot be reduced below $\bar{x}_2$ "in the short run".
- The short-run cost of using the factor combination $(x_1, \bar{x}_2)$ is:

$$w_1 x_1 + w_2 \bar{x}_2.$$

- The short-run cost function is:

$$C_s(y, \bar{x}_2) := \min_{x_1 \in \mathbb{R}_+} \{w_1 x_1 + w_2 \bar{x}_2 : f(x) \geq y\}.$$

Problem

$f(x_1, x_2) = x_1^{\frac{1}{3}} x_2,$

short-run cost function $C_s(y, \bar{x}_2)$*?*

Fixed and variable cost

Definition

Let $C_s : \mathbb{R}_+ \to \mathbb{R}_+$ be a short-run cost function $\Rightarrow$

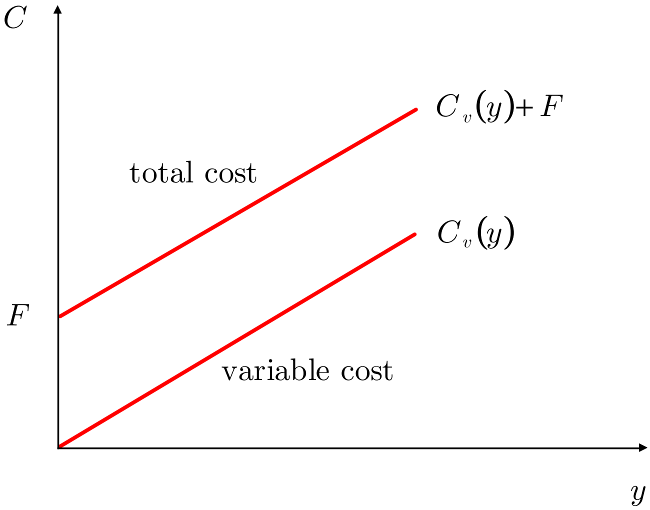
$$F := C_s(0)$$

– fixed cost

$$C_v(y) := C_s(y) - F$$

– the variable cost of producing y.

Fixed and variable cost



Quasi-fixed cost

Definition

Let $C : \mathbb{R}_+ \to \mathbb{R}_+$ be a cost function that is not continuous at 0. In case of $C(0) = 0$ and $\lim_{\substack{y \to 0, \\ y > 0}} C(y) > 0$,

$$F_q := \lim_{\substack{y \to 0, \\ y > 0}} C(y)$$

– quasi-fixed cost.

1. Revisiting the production set
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Profit maximization (output space)

Firm's profit in output space

Definition

Let $C : \mathbb{R}_+ \to \mathbb{R}_+$ be a cost function $\Rightarrow$

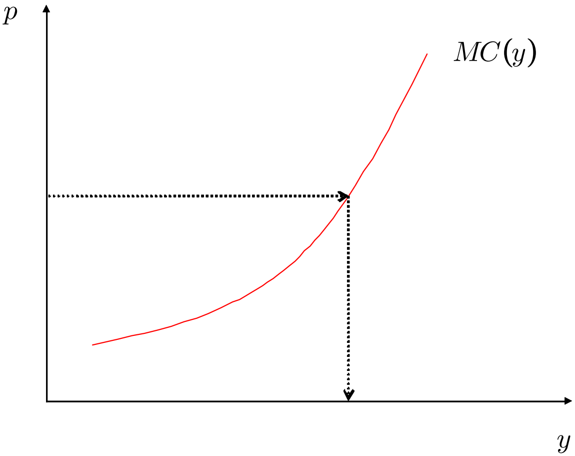
$$\begin{aligned} \Pi &: \mathbb{R}_+ \to \mathbb{R} \text{ and} \\ \underbrace{\Pi(y)}_{\text{profit}} &:= \underbrace{py}_{\text{revenue}} - \underbrace{C(y)}_{\text{cost}} \end{aligned}$$

– a firm's profit in output space.

FOC for profit maximization:

$$MC \overset{!}{=} p.$$

In principle, the marginal-cost curve is the supply curve.



Profit maximization (output space)

Firm's supply function II

Definition

Let $C : \mathbb{R}_+ \to \mathbb{R}_+$ be a (short-run or long-run) cost function $\Rightarrow$

$$\begin{aligned} S &: \mathbb{R}_+ \to \mathbb{R}_+, \\ & p \mapsto S(p) := \arg\max_{y \in \mathbb{R}_+} \Pi(y) \end{aligned}$$

– a firm's supply function.

But: if profit is negative at $p = MC$?

Profit maximization (output space)

Firm's supply function III

- For the short-run supply:

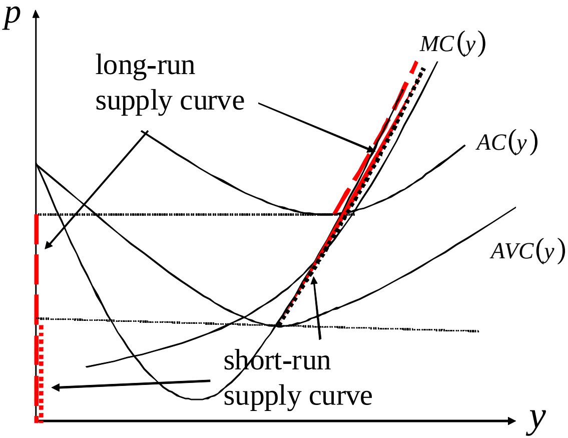
$$\begin{aligned} & S_s(p) = y^* \\ \Rightarrow \quad & \Pi_s(y^*) \geq \Pi_s(0) = -F \\ \Leftrightarrow \quad & py^* - C_v(y^*) - F \geq -F \\ \Leftrightarrow \quad & p \geq \frac{C_v(y^*)}{y^*} =: AVC(y^*) \end{aligned}$$

$AVC(y)$ – average variable cost.

- For the long-run supply:

$$\begin{aligned} & S(p) = y^* \\ \Rightarrow \quad & \Pi(y^*) \geq \Pi(0) = 0 \\ \Leftrightarrow \quad & py^* - C(y^*) \geq 0 \\ \Leftrightarrow \quad & p \geq \frac{C(y^*)}{y^*} = AC(y^*). \end{aligned}$$

Profit maximization (output space)



Profit maximization (output space)

Exercise

Problem

Consider the short-run cost function:

$$C_s(y) := 6y^2 + 15y + 54, y \geq 0$$

and the long-run cost function:

$$C(y) := \begin{cases} 6y^2 + 15y + 54, & y > 0 \\ 0, & y = 0 \end{cases}$$

Determine the short-run and the long-run supply functions S_s and S.

Producer's rent

Definition

The producer's rent at output price $\hat{p}$ is:

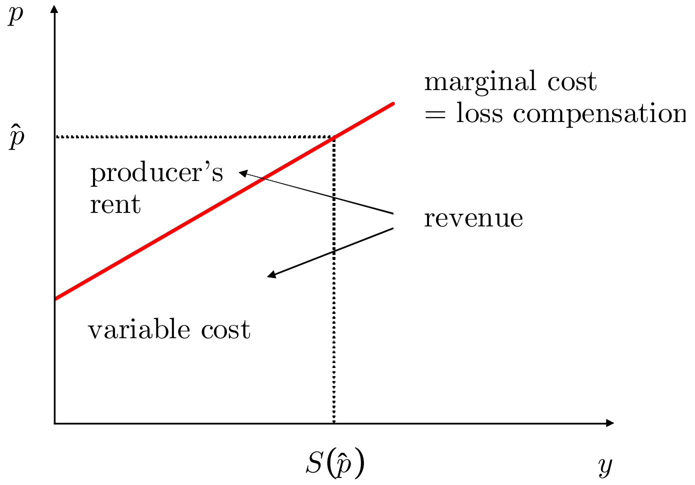
$$\begin{aligned} PR(\hat{p}) &:= CV(0 \to S(\hat{p})) \\ &= \hat{p}S(\hat{p}) - C_v(S(\hat{p})) \\ &= \int_0^{S(\hat{p})} [\hat{p} - MC(X)]\, dX. \end{aligned}$$

Consider

$$\begin{aligned} PR(\hat{p}) &= \hat{p}S(\hat{p}) - C_v(S(\hat{p})) \text{ (definition of producer's rent)} \\ &= \hat{p}S(\hat{p}) - [F + C_v(S(\hat{p}))] + F \text{ (adding } 0 = F - F\text{)} \\ &= \Pi(S(\hat{p})) + F \text{ (definition of profit).} \end{aligned}$$

- fixed cost: producer's rent equals profit plus fixed cost
- no fixed cost: producer's rent equals profit

Producer's rent



Definition

Let $f:\mathbb{R}_+^\ell \to \mathbb{R}_+$ be a production function $\Rightarrow$

$$\begin{aligned} \Pi &: \mathbb{R}_+^\ell \to \mathbb{R} \text{ and} \\ \underbrace{\Pi(x)}_{\text{profit}} &: = \underbrace{pf(x)}_{\text{revenue}} - \underbrace{w \cdot x}_{\text{cost}} \end{aligned}$$

– a firm's profit in input space.

FOC (partial differentiations):

$$MVP_i(x) := p\ MP_i = p\frac{\partial f}{\partial x_i} \overset{!}{=} w_i,\ i = 1, ..., \ell$$

with MVP_i – factor i's marginal value product at x.

Profit maximization (input space)

Firm's factor demand function

Definition

Let $f : \mathbb{R}_+^\ell \to \mathbb{R}_+$ be a production function $\Rightarrow$

$$\begin{aligned} D \ &: \ \mathbb{R}_+^\ell \to \mathbb{R}_+^\ell, \\ &\quad w \mapsto D(w) := \arg\max_{x \in \mathbb{R}_+^\ell} \Pi(x) \end{aligned}$$

– a firm's factor demand function.

Problem

Can you show that profit maximization implies cost minimization? (Hint: Divide the first-order conditions for profit maximization for two inputs!)

Problem

A farmer has a cow that produces milk according to

$$y_M = f(W, G) = W^{\frac{1}{4}} G^{\frac{1}{4}}$$

where M stands for milk, W for water and G for grass. Determine the farmer's demand function for water.

Profit maximization with risk

All owners want their firm to maximize profit if

- prices are not affected by the firm's input and output choice,
- there is no risk involved and
- managers are fully controllable by owners.

Example

- agents A and B possess a firm;
- investment costs 100 ;
- returns 80 or 110, but probability distributions on returns differ.

	w_1 (80)	w_2 (110)	expected value of investment
A's prob. distribution	$\frac{2}{10}$	$\frac{8}{10}$	$\frac{2}{10} \cdot 80 + \frac{8}{10} \cdot 110 - 100 = 4$
B's prob. distribution	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2} \cdot 80 + \frac{1}{2} \cdot 110 - 100 = -5$

Similarly, different risk attitudes may lead to agents' holding opposing view on investment.

Definition (Arrow security)

Let $W = \{w_1, ..., w_m\}$ be a set of m states of the world.

- The contingent good $i \in \{1, .., m\}$ that pays one Euro in case of state of the world w_i and nothing in other states is called an Arrow security.
- If for each state of the world w_i an Arrow security i can be bought and sold for some given price p_i, we say that financial markets are complete.

Problem

Taking note of A' probability distribution, find his willingness to pay for the Arrow security 1. *Hint: Calculate the expected gain from one unit of Arrow security* 1.

Profit maximization with risk

Arrow security

Assume $m = 2$. Arbitrage opportunity presented by

- $p_1 + p_2 < 1$ —> buy both Arrow securities!
- $p_1 + p_2 > 1$ —> sell both Arrow securities!

Definition

Let $W = \{w_1, ..., w_m\}$ be a set of m states of the world with complete financial markets. The Arrow securities are said to be priced correctly (in equilibrium) if $\sum_{i=1}^{m} p_i = 1$ holds.

- Thus, the prices of Arrow securities share the properties of probability distributions.
- Idea: The owners substitute the prices of the Arrow securities for their own probabilities. Then, their different beliefs are unimportant.

Profit maximization with risk

Arrow security takes away risk

The owners A and B consider the investment package:

- spend 100 Euros on the investment,
- buy 20 units of Arrow good 1 and
- sell 10 units of Arrow good 2.

State of the world w_1 : $\underbrace{80}_{\substack{\text{revenue}\\ \text{from}\\ \text{investment}}} + \underbrace{20}_{\substack{\text{revenue}\\ \text{from}\\ \text{Arrow good 1}}} = 100$

State of the world w_2: $\underbrace{110}_{\substack{\text{revenue}\\ \text{from}\\ \text{investment}}} + \underbrace{-10}_{\substack{\text{revenue}\\ \text{from}\\ \text{Arrow good 2}}} = 100$

Therefore, the two owners are indifferent between the two cases. The Arrow securities take all the risk away from them.

Profit maximization with risk

Arrow security

In case of w_1, the investment package is profitable iff

$$
\begin{aligned}
0 &< \underbrace{80}_{\substack{\text{revenue}\\ \text{from}\\ \text{investment}}} - \underbrace{\left(100 + p_1 20 - p_2 10\right)}_{\substack{\text{cost of}\\ \text{investment}\\ \text{package}}} + \underbrace{20}_{\substack{\text{revenue}\\ \text{from}\\ \text{Arrow good 1}}} \\
&= -p_1 20 + p_2 10 \\
&= -p_1 20 + p_2 10 - 100 \underbrace{\left(1 - p_1 - p_2\right)}_{0} \\
&= p_1 80 + p_2 110 - 100.
\end{aligned}
$$

holds, i.e., iff the investment's expected payoff (where Arrow prices take the role of probabilities) is positive.

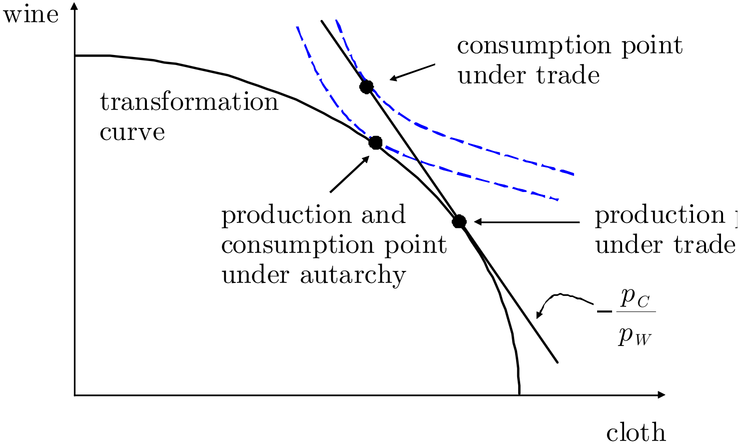
Problem

Show that the profitability of the investment package is equivalent to the investment criterion for state of the world 2, *also.*

The separation function of markets

Examples where markets allow a separation:

- If markets for consumption goods exist, a household with an endowment can consume his endowment but also any other bundle which does not cost more.
- If a market for manager effort is available, an owner-manager can buy additional (on top of his own) manager effort for his firm or supply effort to other firms.
- If Arrow securities are correctly priced, the profitability of an investment decision can be assessed separately from the beliefs and risk attitudes of the several owners.
- International trade allows an economy to consume a bundle different from the bundle produced.


wine
transformation curve
consumption point under trade
production and consumption point under autarchy
production p
under trade
$-\frac{p_C}{p_W}$
cloth

Problem 1

One firm with two factories

		1. unit	2. unit	3. unit	4. unit
marginal	factory 1	2	3	4	5
cost	factory 2	4	5	6	7

How to distribute 2 units among the two factories, how 4 units?

Problem 2

$y = f(x_1, x_2) = x_1^{\frac{1}{2}} x_2$

short-run input $x_2 = 50$

$w_1 = 250,\ w_2 = 3$

short-run marginal cost function SMC?

Problem 3

$y = f(x_1, x_2) = x_1^{\frac{1}{2}} x_2^{\frac{1}{2}}, w_1 = w_2 = 1$

a) Assume $\overline{x_2} = 1$. Sketch

- short-run variable average cost,
- short-run average cost,
- short-run marginal cost

and find the short-run supply curve!

b) Find the long-run supply curve!

Further exercises III

Problem 4

A firm has two factories A and B that obey the production functions $f_A(x_1, x_2) = x_1 \cdot x_2$ and $f_B(x_1, x_2) = x_1 + x_2$, respectively. Given the factor-price ratio $w = \frac{w_1}{w_2}$, how to distribute output in order to minimize costs?

Hints:

- You are free to assume $w_1 \leq w_2$.
- Find the cost functions for the two factories.
- Are the marginal-cost curves upward-sloping or downward-sloping?