Advanced Microeconomics

Production theory

Harald Wiese

University of Leipzig

Part B. Household theory and theory of the firm

1. The household optimum
2. Comparative statics and duality theory
3. **Production theory**
4. Cost minimization and profit maximization

Production theory

Overview

1. The production set
2. Efficiency
3. Exploring the production mountain (function)
4. Edgeworth box and transformation curve
5. Convex production sets and convave production functions

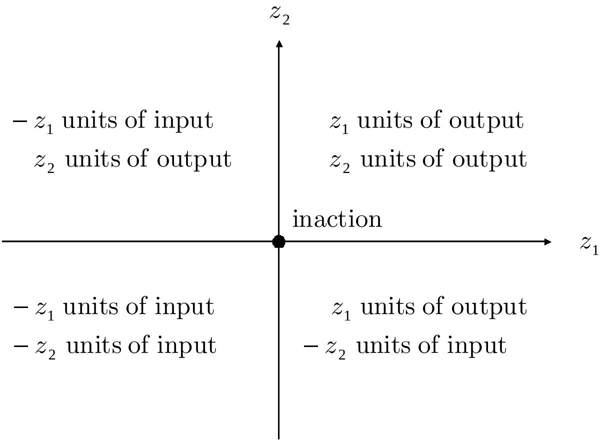
The vector space of goods and inputs

Set of goods bundles:

$$\mathbb{R}^{\ell} := \{(z_1, ..., z_\ell) : z_g \in \mathbb{R}, g = 1, ..., \ell\}.$$

- we allow for $z_g < 0$;
- goods of a negative amount – input or factors of production;
- goods of a positive amount – output or produced goods.

The vector space of goods and inputs



Definition of a production set

Definition

A production set $Z \subseteq \mathbb{R}^\ell$ is the set of input-output combinations such that:

- Z is nonempty,
- Z is closed,
- for every bundle of inputs $(z_1, ..., z_m) \in \mathbb{R}^m_-$, there is a bundle of outputs $(z_{m+1}, ..., z_\ell) \in \mathbb{R}^{\ell-m}_+$ such that:

$$\left(\underbrace{z_1, ..., z_m}_{\text{inputs}}, \underbrace{z_{m+1}, ..., z_\ell}_{\text{outputs}} \right) \in Z$$

  holds;

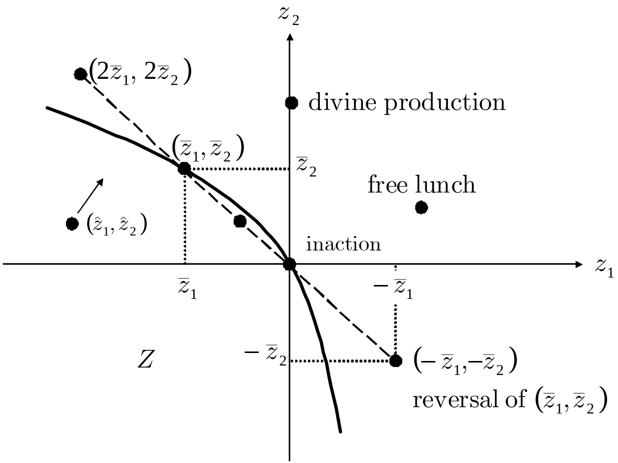
Definition of a production set

Definition

- $\left\{ \left(\underbrace{z_{m+1}, ..., z_\ell}_{\text{outputs}} \right) \in \mathbb{R}_+^{\ell-m} : \left(\underbrace{z_1, ..., z_m}_{\text{inputs}}, \underbrace{z_{m+1}, ..., z_\ell}_{\text{outputs}} \right) \in Z \right\}$ is bounded for every input bundle $\left(\underbrace{z_1, ..., z_m}_{\text{inputs}} \right) \in \mathbb{R}_-^m$,
- Z does not contain any element $z > 0$ and
- $z \in Z$ implies $-z \notin Z$.

The elements in Z – production vectors, production plans or input-output vectors.

Definition of a production set



Divine production: Then let us all with one accord sing praises to our heavenly Lord, who hath made heaven and earth from naught ...

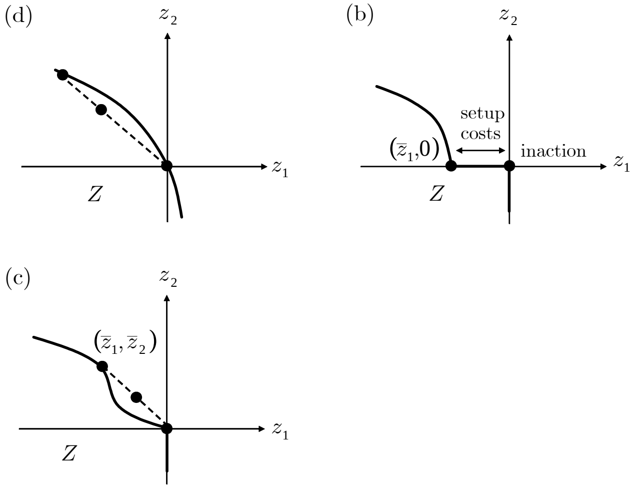
Further axioms

Definition

A production set $Z \subseteq \mathbb{R}^\ell$ obeys

- the possibility of inaction if $0 \in Z$ holds,
- the property of free disposal if $z \in Z$ and $z' \leq z$ implies $z' \in Z$,
- nonincreasing returns to scale if $z \in Z$ implies $kz \in Z$ for all $k \in [0, 1]$,
- nondecreasing returns to scale if $z \in Z$ implies $kz \in Z$ for all $k \geq 1$,
- Z-convexity if Z is convex.

Further axioms



Nonincreasing returns to scale are violated in (b) and (c).

Returns to scale

- Returns to scale are:
  - nonincreasing if production can be scaled down;
  - nondecreasing if production can be scaled up.
- Z-convexity and possibility of inaction imply nonincreasing returns to scale.

Production theory

Overview

1. The production set
2. **Efficiency**
3. Exploring the production mountain (function)
4. Edgeworth box and transformation curve
5. Convex production sets and convave production functions

Definition

Let $Z \subseteq \mathbb{R}^\ell$ be a production set. A point

$$z = \left(\underbrace{z_1, ..., z_m}_{\text{inputs}}, \underbrace{z_{m+1}, ..., z_\ell}_{\text{outputs}} \right)$$

is not input-efficient if another input-output vector

$$\hat{z} = \left(\underbrace{\hat{z}_1, ..., \hat{z}_m}_{\text{inputs}}, \underbrace{z_{m+1}, ..., z_\ell}_{\text{outputs}} \right)$$

exists such that $(\hat{z}_1, ..., \hat{z}_m) > (z_1, ..., z_m)$.
$\hat{z}$ – an input improvement over z.

Definition

Let $Z \subseteq \mathbb{R}^\ell$ be a production set. A point

$$z = \left(\underbrace{z_1, ..., z_m}_{\text{inputs}}, \underbrace{z_{m+1}, ..., z_\ell}_{\text{outputs}} \right)$$

is not output-efficient if another input-output vector

$$\hat{z} = \left(\underbrace{z_1, ..., z_m}_{\text{inputs}}, \underbrace{\hat{z}_{m+1}, ..., \hat{z}_\ell}_{\text{outputs}} \right)$$

exists such that $(\hat{z}_{m+1}, ..., \hat{z}_\ell) > (z_{m+1}, ..., z_\ell)$.
$\hat{z}$ – an output improvement over z.

Input efficiency and output efficiency

Improvement

Definition

Let $Z \subseteq \mathbb{R}^\ell$ be a production set. A point

$$z = \left(\underbrace{z_1, ..., z_m}_{\text{inputs}}, \underbrace{z_{m+1}, ..., z_\ell}_{\text{outputs}} \right)$$

is not efficient if another input-output vector

$$\hat{z} = \left(\underbrace{\hat{z}_1, ..., \hat{z}_\ell}_{\text{inputs and outputs}} \right)$$

exists such that $\hat{z} > z$ holds.

$\hat{z}$ – an improvement over z.

1. The production set
2. Efficiency
3. **Exploring the production mountain (function)**
4. Edgeworth box and transformation curve
5. Convex production sets and convave production functions

Production function and isoquant

Definition

Let $Z \subseteq \mathbb{R}^\ell$ be a production set. A function $f : \mathbb{R}_+^{\ell-1} \to \mathbb{R}_+$ defined by

$$f(x_1, ..., x_{\ell-1}) = \max \{ y \in \mathbb{R}_+ : (-x_1, ..., -x_{\ell-1}, y) \in Z \}.$$

– the production function for y.

Problem

Find the production functions for the production set

$$Z = \left\{ (z_1, z_2) \in \mathbb{R}^2 \mid z_2 \leq -(z_1)^2 \text{ if } z_1 \geq 0 \text{ and } z_2 \leq -\frac{1}{2} z_1 \text{ if } z_1 < 0 \right\}$$

Production function and isoquant

Definition

Let f be a production function on $\mathbb{R}_+^{\ell-1}$.

$$B_{\hat{x}} := \left\{ x \in \mathbb{R}_+^{\ell-1} : f(x) \geq f(\hat{x}) \right\}$$

– the better set $B_{\hat{x}}$ of $\hat{x}$;

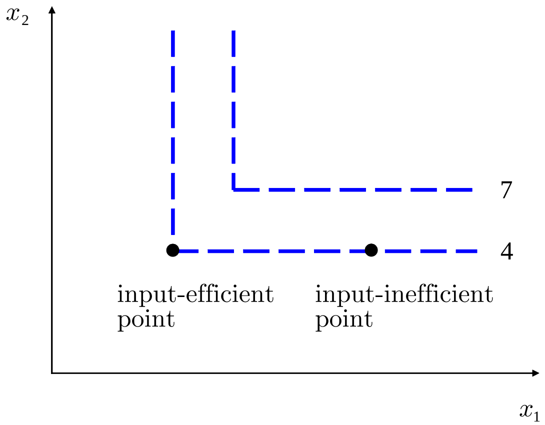
$$W_{\hat{x}} := \left\{ x \in \mathbb{R}_+^{\ell-1} : f(x) \leq f(\hat{x}) \right\}$$

– the worse set $W_{\hat{x}}$ of $\hat{x}$;

$$I_{\hat{x}} := B_{\hat{x}} \cap W_{\hat{x}} = \left\{ x \in \mathbb{R}_+^{\ell-1} : f(x) = f(\hat{x}) \right\}$$

– $\hat{x}$'s isoquant $I_{\hat{x}}$.

Production function and isoquant



Definition

A production function f obeys:

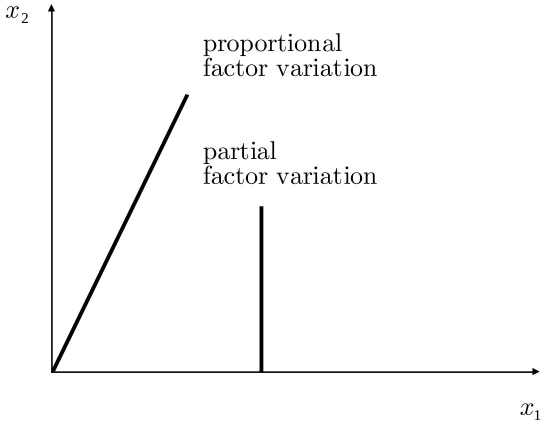
- weak monotonicity iff $x > x'$ implies $f(x) \geq f(x')$,
- strict monotonicity iff $x > x'$ implies $f(x) > f(x')$, and
- local non-satiation at x' iff a bundle x with $f(x) > f(x')$ can be found in every ε-ball with center x'.

Cardinality of production functions vs ordinality of preferences!

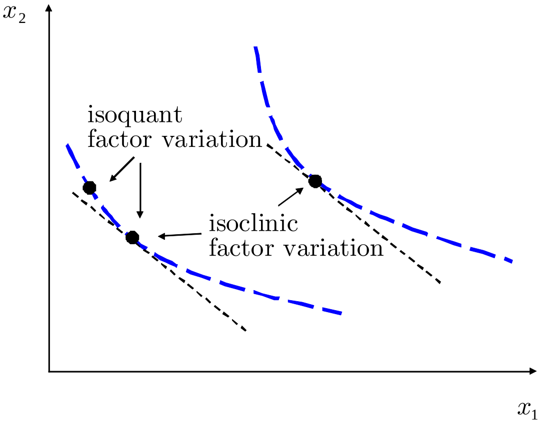
Factor variations

- Partial factor variation: We change one factor only and keep the other factors constant.
- Proportional factor variation: We change all the factors while keeping proportions constant.
- Isoquant factor variation: We change the factors so as to keep output constant.
- Isoclinic factor variation: We change the factors so as to keep the marginal rate of technical substitution constant.

Partial and proportional factor variations



Isoquant and isoclinic factor variation



Partial factor variation

- The marginal productivity of factor i:

$$MP_i := \frac{\partial f}{\partial x_i}.$$

- Average productivity of factor i:

$$AP_i := \frac{f(x_i)}{x_i}.$$

Problem

Suggest a definition of production elasticity. Do you see how the production elasticity depends on the marginal and the average productivity?

Problem

Calculate factor 1's production elasticity for the Cobb-Douglas production function f given by $f(x_1, x_2) = x_1^a x_2^b$, $a, b \geq 0$.

Marginal something equals average something

Lemma

Let $f : \mathbb{R} \to \mathbb{R}$ *be any differentiable (production) function* $\Rightarrow$
$\frac{df}{dx}\Big|_{x=0} = \frac{f(x)}{x}\Big|_{x=0}$ *if* $f(0) = 0$ *holds.*

Not difficult to show.

Examples

Average product equals marginal product for the first „very small" unit, price equals marginal revenue for the first „very small" unit.

Marginal something equals average something

Lemma

Let $f : \mathbb{R} \to \mathbb{R}$ be any differentiable (production) function. Assume $x > 0$ $\Rightarrow$

$$\frac{df}{dx} > \frac{f(x)}{x} \Leftrightarrow \frac{d\frac{f(x)}{x}}{dx} > 0.$$

Problem

Provide a proof by applying the quotient rule of differentiation to $\frac{d\frac{f(x)}{x}}{dx}$.

Marginal something equals average something

Summary

- If the marginal productivity is above the average productivity, the average productivity increases.
- If the marginal productivity equals the average productivity, the average productivity is constant.

This holds for:
– marginal revenue and average revenue (price),
– marginal cost and average cost and
– marginal profit and average profit.

Proportional factor variation: returns to scale

Definition

Proportional factor variation:

$$(x_1, ..., x_\ell) \mapsto t\,(x_1, ..., x_\ell) = (tx_1, ..., tx_\ell)\,.$$

with

- x – the factors of production and
- t – scalar.

Proportional factor variation: returns to scale

Definition

A production function $f : \mathbb{R}_+^\ell \to \mathbb{R}_+$ is characterized

- by constant returns to scale if
$$f(tx) = tf(x) \text{ for all } t \geq 0;$$
- by increasing returns to scale if
$$f(tx) \geq tf(x) \text{ for all } t \geq 1;$$
- by decreasing returns to scale if
$$f(tx) \leq tf(x) \text{ for all } t \geq 1$$

hold for all $x \in \mathbb{R}_+^\ell$, respectively.

Returns to scale

Scale elasticity

Definition

Let $f : \mathbb{R}_+^\ell \to \mathbb{R}_+$ be a production function. The scale elasticity at $x = (x_1, ..., x_\ell)$ is:

$$\varepsilon_{y,t} = \left. \frac{\frac{df(tx)}{f(tx)}}{\frac{dt}{t}} \right|_{t=1} = \left. \frac{df\,(tx)}{dt} \frac{t}{f\,(tx)} \right|_{t=1}.$$

Lemma

We have

- *increasing returns to scale at* $x \in \mathbb{R}_+^\ell$ *iff* $\varepsilon_{y,t} \geq 1$ *holds,*
- *decreasing returns to scale at* $x \in \mathbb{R}_+^\ell$ *in case of* $\varepsilon_{y,t} \leq 1$ *and*
- *constant returns to scale at* $x \in \mathbb{R}_+^\ell$ *iff* $\varepsilon_{y,t} = 1$ *is true.*

Returns to scale

Problem

Calculate the scale elasticity for the Cobb-Douglas production function f given by $f(x_1, x_2) = x_1^a x_2^b$, $a, b \geq 0$.

Isoquant factor variation: Marginal rate of technical substitution

Definition

If the function I_y is differentiable and if the production function is monotonic,

$$MRTS = \left| \frac{dI_y(x_1)}{dx_1} \right|$$

– the marginal rate of technical substitution between factor 1 and factor 2 (or of factor 2 for factor 1).

Lemma

Let f be a differentiable production function $\Rightarrow$

$$MRTS(x_1) = \frac{\frac{\partial f}{\partial x_1}}{\frac{\partial f}{\partial x_2}}.$$

Efficiency requires:

$$MRTS^A \overset{!}{=} MRTS^B$$

Example

$$(3 =) \left| \frac{dx_2^A}{dx_1^A} \right| = MRTS^A < MRTS^B = \left| \frac{dx_2^B}{dx_1^B} \right| (= 5)$$

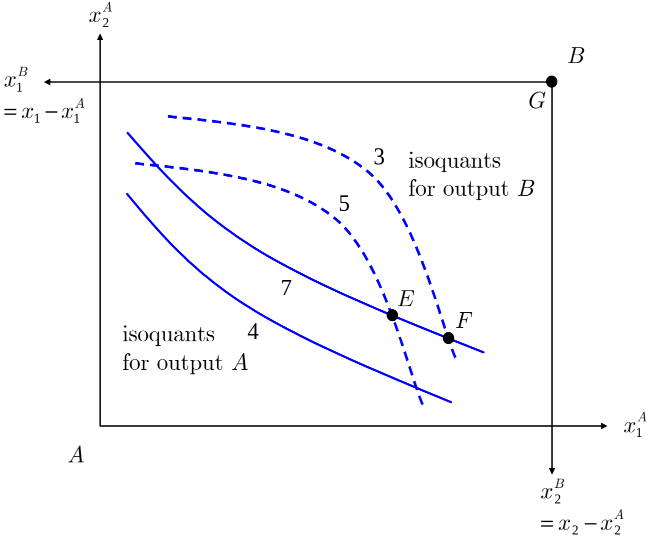
Production theory

Overview

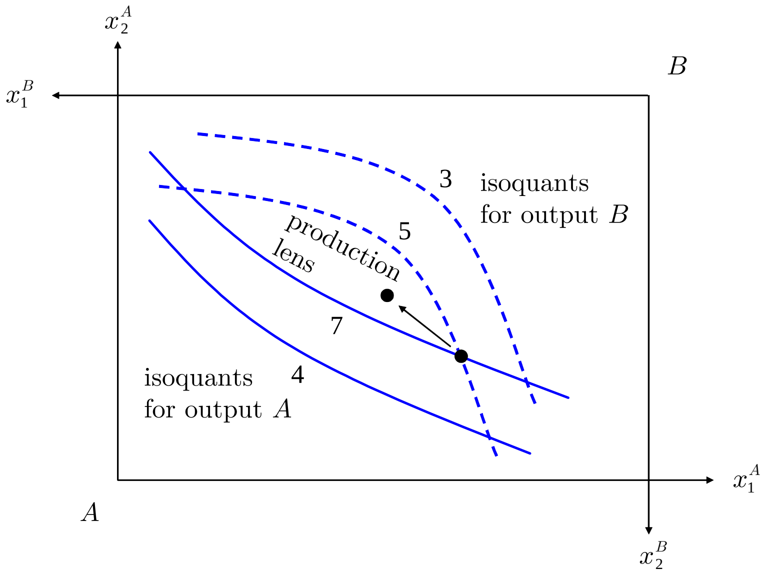
1. The production set
2. Efficiency
3. Exploring the production mountain (function)
4. **Edgeworth box and transformation curve**
5. Convex production sets and convave production functions

Edgeworth box

If inputs are attributable to specific outputs:



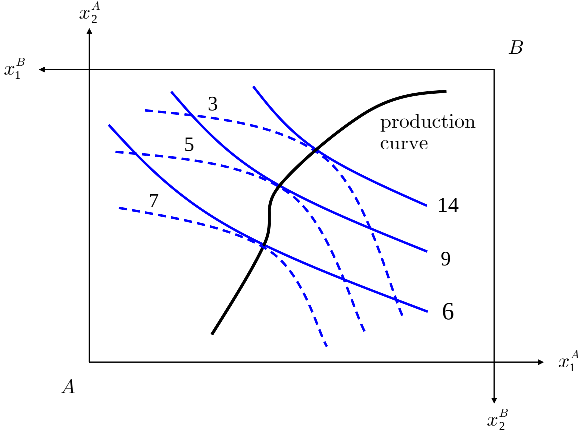
Edgeworth box



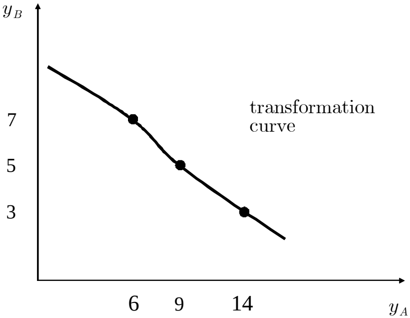
Production curve

Definition

Production curve – the locus of all the points of tangency between two isoquants.



Transformation curve (production-possibility frontier)



Problem

Using a transformation curve, discuss output efficiency.

Problem

Production curve and transformation curve for
$x_1 = x_2 = 100, y_A = x_1^A + x_2^A$ *and* $y_B = \left(x_1^B\right)^{\frac{1}{2}} \left(x_2^B\right)^{\frac{1}{2}}$

Definition

Assume that the transformation curve defines a differentiable function $y_A \mapsto y_B$.

$$MRT := \left| \frac{dy_B}{dy_A} \right|$$

– the marginal rate of transformation between good A and good B.

1. The production set
2. Efficiency
3. Exploring the production mountain (function)
4. Edgeworth box and transformation curve
5. **Convex production sets and convave production functions**

Convexity of the production set and concavity of the production function

Lemma

Let

- Z *be a production set where the first* $\ell - 1$ *entries are always nonpositive;*
- f *be the production function associated with* Z*;*
- Z *obey free disposal.*

$\Rightarrow$ Z *is convex iff the corresponding production function* f *is concave.*

See manuscript.

Convex production sets versus convex better sets

Example

Consider the following production function:

$$f(x,y) = xy.$$

- It obeys strict quasi-concavity (Cobb-Douglas preferences!).
- It is not concave:

$$\begin{aligned} f(k(0,0)+(1-k)(1,1)) &= f(1-k,1-k) = (1-k)^2 < 1-k \\ &= k\cdot 0 + (1-k)\cdot 1 \\ &= kf(0,0)+(1-k)f(1,1). \end{aligned}$$

for $0 < k < 1$.

Problem

Show that every concave function is quasi-concave.
Remember:
$f : \mathbb{R}^\ell \to \mathbb{R}$ *is quasi-concave if*

$$f\left(kx + (1-k)\,x'\right) \geq \min\left(f\left(x\right), f\left(x'\right)\right)$$

holds for all $x,\ x' \in \mathbb{R}^\ell$ *and all* $k \in [0,1]$.
$f : \mathbb{R}^\ell \to \mathbb{R}$ *is concave if*

$$f\left(kx + (1-k)\,x'\right) \geq kf\left(x\right) + (1-k)\,f\left(x'\right)$$

holds for all $x, x' \in \mathbb{R}^\ell$ *and for all* $k \in [0,1]$.

Convex production sets versus convex better sets

Lemma

Let f be a continuous production function on $\mathbb{R}_+^\ell$ $\Rightarrow$

f's production set under free disposal strictly convex	$\Longleftrightarrow$	f strictly concave	$\Longrightarrow$	f concave	$\Longleftrightarrow$	f's production set under free disposal convex
		$\Downarrow$		$\Downarrow$		$\Downarrow$
f's better sets strictly convex	$\Longleftarrow$	f strictly quasi-concave	$\Longrightarrow$	f quasi-concave	$\Longleftrightarrow$	f's better sets convex
		$\Updownarrow$		$\Updownarrow$		
f's better sets strictly convex and local nonsatiation	$\Longrightarrow$	f's isoquants strictly concave	$\Longrightarrow$	f's isoquants concave		

What about concave utility functions?

There are functions that are not concave but still quasi-concave:

Example

Consider the utility functions U and V given by $U(x,y) = xy$ and $V(x,y) = x^{\frac{1}{3}} y^{\frac{1}{3}}$. We can apply the increasing function $\tau : \mathbb{R} \to \mathbb{R}$ given by $\tau(U) = U^{\frac{1}{3}}$ and obtain

$$\begin{aligned} (\tau \circ U)(x,y) &= \tau(U(x,y)) \\ &= \tau(xy) \\ &= (xy)^{\frac{1}{3}} \\ &= V(x,y) \end{aligned}$$

U and V represent the same preferences, but

- U is neither convex nor concave but quasi-concave;
- V is concave $\Rightarrow$ V is quasi-concave.

Further exercises

Problem 1

Sketch a few isoquants that reflect decreasing returns to scale.

Problem 2

Determine the production set for the production function
$y = f(x_1, x_2) = \min\{x_1, x_2\}, \ x_1, x_2 \geq 0.$

Further exercises

Problem 3
Let f be a homogeneous function of degree λ (i.e., $f(tx) = t^{\lambda} \cdot f(x)$).
Show

$$\sum_i \frac{\partial f}{\partial x_i} x_i = \lambda t^{\lambda-1} f(x)$$

and, for $\lambda = 1$, Euler's theorem,

$$\sum_i \frac{\partial f}{\partial x_i} x_i = f(x).$$

Hint: Calculate $\frac{\partial f(tx)}{\partial t}$ *and* $\frac{\partial\left[t^{\lambda} f(x)\right]}{\partial t}$.