Advanced Microeconomics

Comparative statics and duality theory

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Part B. Household theory and theory of the firm

1. The household optimum
2. **Comparative statics and duality theory**
3. Production theory
4. Cost minimization and profit maximization

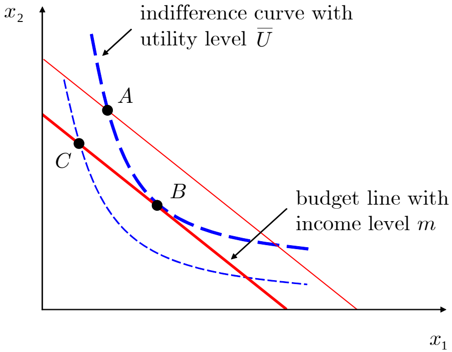
Comparative statics and duality theory

Overview

1. The duality approach
2. Shephard's lemma
3. The Hicksian law of demand
4. Slutsky equations
5. Compensating and equivalent variations

Maximization and minimization problem

- Maximization problem: Find the bundle that maximizes the utility for a given budget line.



- Minimization problem: Find the bundle that minimizes the expenditure needed to achieve a given utility level.

The expenditure function and the Hicksian demand function I

Expenditure function:

$$\begin{aligned} e \quad : \quad & \mathbb{R}^\ell \times \mathbb{R} \to \mathbb{R}, \\ & \left(p, \bar{U}\right) \mapsto e\left(p, \bar{U}\right) := \min_{\substack{x \text{ with} \\ U(x) \geq \bar{U}}} px \end{aligned}$$

The solution to the minimization problem is called the Hicksian demand function:

$$\begin{aligned} \chi \quad : \quad & \mathbb{R}^\ell \times \mathbb{R} \to \mathbb{R}^\ell_+, \\ & \left(p, \bar{U}\right) \mapsto \chi\left(p, \bar{U}\right) := \arg \min_{\substack{x \text{ with} \\ U(x) \geq \bar{U}}} px \end{aligned}$$

The expenditure function and the Hicksian demand function II

Problem

Express

- *e in terms of* χ *and*
- *V in terms of the household optima!*

Lemma

For any $\alpha > 0$*:*

$$\chi\left(\alpha p, \bar{U}\right) = \chi\left(p, \bar{U}\right) \text{ and } e\left(\alpha p, \bar{U}\right) = \alpha e\left(p, \bar{U}\right).$$

Obvious?!

The expenditure function and the Hicksian demand function III

Problem

Determine the expenditure function and the Hicksian demand function for the Cobb-Douglas utility function $U(x_1, x_2) = x_1^a x_2^{1-a}$ *with* $0 < a < 1$*! Hint: We know that the indirect utility function* V *is given by:*

$$\begin{aligned} V(p, m) &= U(x(p, m)) \\ &= \left(a\frac{m}{p_1}\right)^a \left((1-a)\frac{m}{p_2}\right)^{1-a} \\ &= \left(\frac{a}{p_1}\right)^a \left(\frac{1-a}{p_2}\right)^{1-a} m. \end{aligned}$$

Hicksian demand and the expenditure function

	utility maximization	expenditure minimization
objective function	utility	expenditure
parameters	prices p, income m	prices p, utility $\bar{U}$
notation for best bundle(s)	$x(p, m)$	$\chi(p, \bar{U})$
name of demand function	Marshallian	Hicksian
value of objective function	$V(p, m) = U(x(p, m))$	$e(p, \bar{U}) = p \cdot \chi(p, \bar{U})$

Applying the Lagrange method (recipe)

$$L(x,\mu) = \sum_{g=1}^{\ell} p_g x_g + \mu\left[\bar{U} - U(x)\right]$$

with:

- U – strictly quasi-concave and strictly monotonic utility function;
- prices $p >> 0$;
- $\mu > 0$ – Lagrange multiplier - translates a utility surplus (in case of $U(x) > \bar{U}$) into expenditure reduction
- Increasing consumption has
  - a positive direct effect on expenditure, but
  - a negative indirect effect via a utility surplus (scope for expenditure reduction) and μ

Applying the Lagrange method (recipe)

Differentiate L with respect to x_g :

$$\frac{\partial L\left(x_1, x_2, ..., \mu\right)}{\partial x_g} = p_g - \mu \frac{\partial U\left(x_1, x_2, ..., x_\ell\right)}{\partial x_g} \overset{!}{=} 0 \text{ or } \frac{\partial U\left(x_1, x_2, ..., x_\ell\right)}{\partial x_g} \overset{!}{=}$$

and hence, for two goods g and k

$$\frac{\frac{\partial U(x_1, x_2, ..., x_\ell)}{\partial x_g}}{\frac{\partial U(x_1, x_2, ..., x_\ell)}{\partial x_k}} \overset{!}{=} \frac{p_g}{p_k} \text{ or } MRS \overset{!}{=} MOC$$

Note also: $\frac{\partial L(x,\mu)}{\partial \mu} = \bar{U} - U\left(x\right) \overset{!}{=} 0$

Applying the Lagrange method

Comparing the Lagrange multipliers

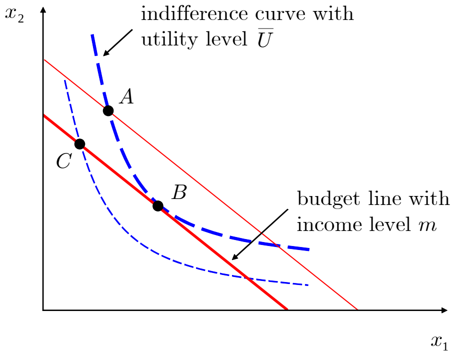
- λ – the shadow price for utility maximization (translates additional income m into higher utility: $\lambda = \frac{\partial V}{\partial m}$);
- μ – the shadow price for expenditure minimization (translates additional utility $\bar{U}$ into higher expenditure: $\mu = \frac{\partial e(p,\bar{U})}{\partial \bar{U}}$).

We note without proof

$$\mu = \frac{1}{\lambda}$$

The duality theorem

here, duality does work

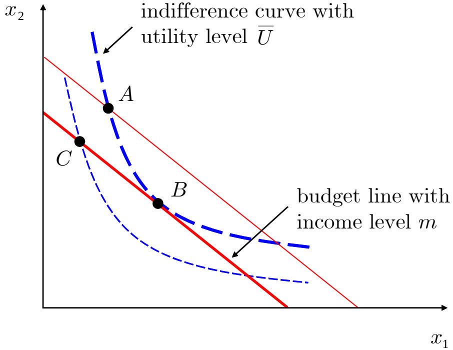


- At the budget line through point C (budget m), the household optimum is at point B and the utility is $U(B) = V(p, m)$.
- The expenditure needed to obtain B's utility level or A's utility level is equal to m:

$$e(p, V(p, m)) = m.$$

The duality theorem

here, duality does work



- The minimal expenditure for the indifference curve passing through A (utility level $\bar{U}$) is denoted by $e\left(p, \bar{U}\right)$ and achieved by bundle B.
- With income $e\left(p, \bar{U}\right)$ (at point C), the highest achievable utility is $V\left(p, e\left(p, \bar{U}\right)\right) = \bar{U}$.

The duality theorem

Conditions for duality

Theorem

Let $U : \mathbb{R}^{\ell}_{+} \to \mathbb{R}$ be a continuous utility function that obeys local nonsatiation and let $p >> 0$ be a price vector. $\Rightarrow$

- *If $x(p, m)$ is the household optimum for $m > 0$:*

$$\chi(p, V(p, m)) = x(p, m)$$

$$e(p, V(p, m)) = m.$$

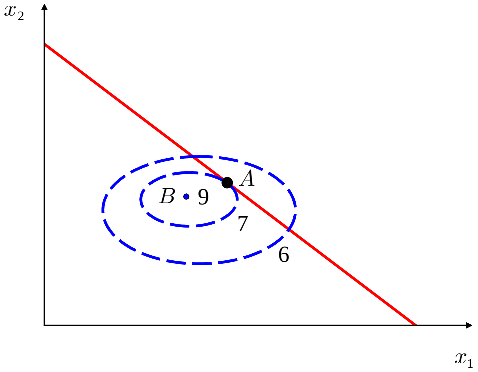
- *If $\chi(p, \bar{U})$ is the expenditure-minimizing bundle for $\bar{U} > U(0)$:*

$$x(p, e(p, \bar{U})) = \chi(p, \bar{U})$$

$$V(p, e(p, \bar{U})) = \bar{U}.$$

The duality theorem

here, duality does not work



Here, local nonsatiation is violated.

- At (p, m), the household optimum is at the bliss point with $V(p, m) = 9$.
- The expenditure needed to obtain this utility level is smaller than m:

$$e(p, V(p, m)) < m.$$

Main results

Theorem

Consider a household with a continuous utility function U, Hicksian demand function χ and expenditure function e.

- *Shephard's lemma: The price increase of good g by one small unit increases the expenditure necessary to uphold the utility level by χ_g.*
- *Roy's identity: A price increase of good g by one small unit decreases the budget available for the other goods by χ_g and indirect utility by the product of the marginal utility of income $\frac{\partial V}{\partial m}$ and χ_g.*
- *Hicksian law of demand: If the price of a good g increases, the Hicksian demand χ_g does not increase.*
- *The Hicksian cross demands are symmetric:* $\frac{\partial \chi_g(p,\bar{U})}{\partial p_k} = \frac{\partial \chi_k(p,\bar{U})}{\partial p_g}$.
- *Slutsky equations*
  see next slide

Main results

Theorem

- *Money-budget Slutsky equation:*

$$\frac{\partial x_g}{\partial p_g} = \frac{\partial \chi_g}{\partial p_g} - \frac{\partial x_g}{\partial m} \chi_g .$$

- *Endowment Slutsky equation:*

$$\frac{\partial x_g^{endowment}}{\partial p_g} = \frac{\partial \chi_g}{\partial p_g} + \frac{\partial x_g^{money}}{\partial m} \left(\omega_g - \chi_g \right) .$$

Shephard's lemma

Overview

1. The duality approach
2. **Shephard's lemma**
3. The Hicksian law of demand
4. Slutsky equations
5. Compensating and equivalent variations

Shephard's lemma

- Assume a price increase for a good g by one small unit.
- To achieve the same utility level, expenditure must be increased by at most

$$\frac{\partial e}{\partial p_g} \leq \chi_g$$

- Shephard's Lemma (see manuscript):

$$\frac{\partial e}{\partial p_g} = \chi_g$$

Roy's identity

Duality equation:

$$\bar{U} = V\left(p, e\left(p, \bar{U}\right)\right).$$

Differentiating with respect to p_g:

$$\begin{aligned} 0 &= \frac{\partial V}{\partial p_g} + \frac{\partial V}{\partial m}\frac{\partial e}{\partial p_g} \\ &= \frac{\partial V}{\partial p_g} + \frac{\partial V}{\partial m}\chi_g \text{ (Shephard's lemma)} \end{aligned}$$

Roy's identity: $\frac{\partial V}{\partial p_g} = \frac{\partial V}{\partial m}\left(-\chi_g\right).$

1. The duality approach
2. Shephard's lemma
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Compensated (Hicksian) law of demand

Assume:

- p and p' – price vectors from $\mathbb{R}^{\ell}$;
- $\chi(p, \bar{U}) \in \mathbb{R}^{\ell}_{+}$ and $\chi(p', \bar{U}) \in \mathbb{R}^{\ell}_{+}$ – expenditure-minimizing bundles necessary to achieve a utility of at least $\bar{U}$.

$$p' \cdot \underbrace{\chi(\bar{U}, p')}_{\substack{\text{expenditure-}\\ \text{minimizing}\\ \text{bundle at } p'}} \leq p' \cdot \underbrace{\chi(\bar{U}, p)}_{\substack{\text{expenditure-}\\ \text{minimizing}\\ \text{bundle at } p}} .$$

... (see the manuscript)

$\Rightarrow$ If the price of one good increases, the Hicksian demand for that good cannot increase: $\frac{\partial \chi_g}{\partial p_g} \leq 0$.

Definition

Let $f : M \rightarrow \mathbb{R}$ be a function on a convex domain $M \subseteq \mathbb{R}^\ell$.

$\Rightarrow$

- f is concave if

$$f(kx + (1-k)y) \geq kf(x) + (1-k)f(y)$$

  for all $x, y \in M$ and for all $k \in [0,1]$ (for $\leq$ – convex).

- f is strictly concave if

$$f(kx + (1-k)y) > kf(x) + (1-k)f(y)$$

  holds for all $x, y \in M$ with $x \neq y$ and for all $k \in (0,1)$ (for $<$ – strictly convex).

Hesse matrix

Definition

Let $f : \mathbb{R}^{\ell} \to \mathbb{R}$ be a function.
The second-order partial derivative of f with respect to x_i and x_j (if it exists) is given by

$$f_{ij}(x) := \frac{\partial \frac{\partial f(x)}{\partial x_i}}{\partial x_j} = \frac{\partial^2 f(x)}{\partial x_i \partial x_j}.$$

If all the second-order partial derivatives exist, the Hesse matrix of f is given by

$$H_f(x) = \begin{pmatrix} f_{11}(x) & f_{12}(x) & & f_{1\ell}(x) \\ f_{21}(x) & f_{22}(x) & & \\ & & & \\ f_{\ell 1}(x) & f_{n2}(x) & & f_{\ell\ell}(x) \end{pmatrix}.$$

Problem

Hesse matrix

Believe me (or check in the script) that

Lemma (diagonal entries)

If *a function* $f : \mathbb{R}^\ell \to \mathbb{R}$ *is concave (strictly concave), the diagonal entries of its Hesse matrix are non-positive (negative).*

Remember: For $f : \mathbb{R} \to \mathbb{R}$, f is concave **iff** $f''(x) \leq 0$ for all $x \in \mathbb{R}$ (chapter on von Neumann-Morgenstern utility)

Lemma (symmetry)

If all the second-order partial derivatives of $f : \mathbb{R}^\ell \to \mathbb{R}$ *exist and are continuous, then*

$$f_{ij}(x) = f_{ji}(x) \text{ for all } i, j = 1, ..., \ell$$

Lemma (expenditure fct. concave)

The expenditure function is concave.

Hesse matrix of expenditure function

$$H_e\left(p,\bar{U}\right)=\begin{pmatrix}\frac{\partial^2 e(p,\bar{U})}{(\partial p_1)^2} & \frac{\partial^2 e(p,\bar{U})}{\partial p_1 \partial p_2} & & \frac{\partial^2 e(p,\bar{U})}{\partial p_1 \partial p_\ell} \\ \frac{\partial^2 e(p,\bar{U})}{\partial p_2 \partial p_1} & \frac{\partial^2 e(p,\bar{U})}{(\partial p_2)^2} & & \\ & & & \\ \frac{\partial^2 e(p,\bar{U})}{\partial p_\ell \partial p_1} & \frac{\partial^2 e(p,\bar{U})}{\partial p_\ell \partial p_2} & & \frac{\partial^2 e(p,\bar{U})}{(\partial p_\ell)^2}\end{pmatrix}.$$

Compensated (Hicksian) law of demand

By Shephard's lemma:

$$\frac{\partial e\left(p, \bar{U}\right)}{\partial p_g} = \chi_g\left(p, \bar{U}\right)$$

Forming the derivative of the Hicksian demand, we find

$$\frac{\partial \chi_g\left(p, \bar{U}\right)}{\partial p_k} = \frac{\partial \frac{\partial e\left(p, \bar{U}\right)}{\partial p_g}}{\partial p_k} = \frac{\partial^2 e\left(p, \bar{U}\right)}{\partial p_g \partial p_k}$$

with two interesting conclusions:

1. $g = k$
   Hicksian law of demand (by lemma "expenditure fct. concave" and lemma "diagonal entries"):

$$\frac{\partial \chi_g\left(p, \bar{U}\right)}{\partial p_g} \leq 0$$

Substitutes and complements (the Hicksian definition)

2. $g \neq k$:
   If the off-diagonal entries in the expenditure function's Hesse matrix are continuous, lemma "symmetry" implies

$$\frac{\partial \chi_g(p, \bar{U})}{\partial p_k} = \frac{\partial \chi_k(p, \bar{U})}{\partial p_g}.$$

Definition

Goods g and k are

- substitutes if

$$\frac{\partial \chi_g(p, \bar{U})}{\partial p_k} = \frac{\partial \chi_k(p, \bar{U})}{\partial p_g} \geq 0;$$

- complements if

$$\frac{\partial \chi_g(p, \bar{U})}{\partial p_k} = \frac{\partial \chi_k(p, \bar{U})}{\partial p_g} \leq 0.$$

Every good has at least one substitute.

For all $\alpha > 0$:

$$\chi_g\left(\alpha p, \bar{U}\right) = \chi_g\left(p, \bar{U}\right).$$

Differentiating with the adding rule (chapter on preferences) yields

$$\sum_{k=1}^{\ell} \frac{\partial \chi_g\left(\alpha p, \bar{U}\right)}{\partial p_k} \cdot p_k = \frac{\partial \chi_g\left(\alpha p, \bar{U}\right)}{\partial \alpha} = \frac{\partial \chi_g\left(p, \bar{U}\right)}{\partial \alpha} = 0$$

$\Rightarrow$

$$\sum_{\substack{k=1,\\ k\neq g}}^{\ell} \frac{\partial \chi_g\left(\alpha p, \bar{U}\right)}{\partial p_k} \cdot p_k = -\underbrace{\frac{\partial \chi_g\left(\alpha p, \bar{U}\right)}{\partial p_g}}_{\leq 0} \cdot p_g \geq 0.$$

$\Rightarrow$

Lemma

Assume $\ell \geq 2$ and $p >> 0$. Every good has at least one substitute.

Slutsky equations

Overview

Three effects of a price increase

$$\frac{\partial \chi_1 \left(p, \bar{U}\right)}{\partial p_1} \underset{!}{\leq} 0 \text{ and } \frac{\partial x_1 \left(p, m\right)}{\partial p_1} \underset{?}{\leq} 0$$

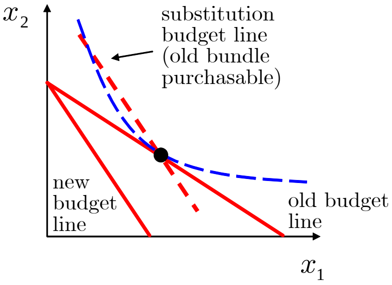
1. Substitution effect or opportunity-cost effect: $p_1 \uparrow$
   - $\Rightarrow p_1 / p_2 \uparrow$
   - $\Rightarrow x_1 \downarrow$ and $x_2 \uparrow$
2. Consumption-income effect: $p_1 \uparrow$
   - $\Rightarrow$ overall consumption possibilities decrease
   - $\Rightarrow x_1 \downarrow$ if 1 is a normal good
3. Endowment-income effect: $p_1 \uparrow$
   - $\Rightarrow$ value of endowment increases
   - $\Rightarrow x_1 \uparrow$ if 1 is a normal good

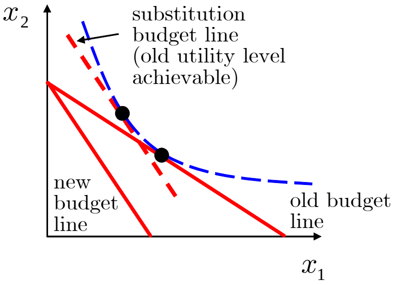
Two different substitution effects

Definitions

In response to a price change, there are two different ways to keep real income constant:

- Old-household-optimum substitution effect
- Old-utility-level substitution effect





Two different substitution effects

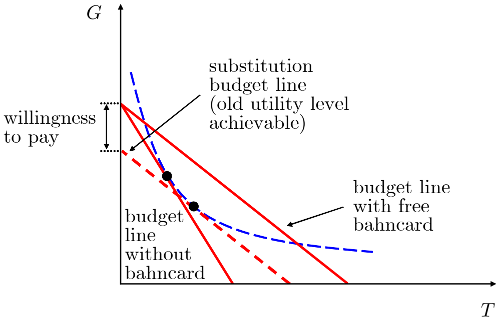
Bahncard 50

Example

Two goods: train rides T and other goods G

- $p_T = 0.2$ (per kilometer), $p_G = 1$.
- "Bahncard 50": p_T reduced to 0.1.

Willingness to pay for the "Bahncard 50"?



Two different substitution effects

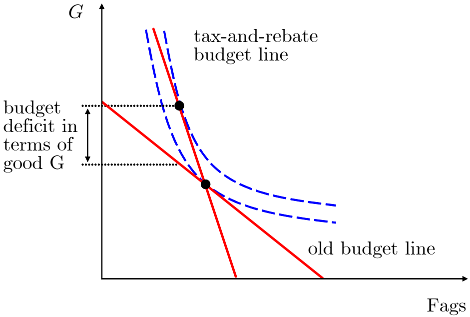
Tax and rebate

Example

You smoke 10 cigarettes per day. The government is concerned about your health.

- Quantity tax of 10 cents, but
- rebate of 1 Euro per day.

Budget deficit in terms of the other goods?



The Slutsky equation for the money budget

Derivation

Duality equation: $\chi_g\left(p, \bar{U}\right) = x_g\left(p, e\left(p, \bar{U}\right)\right)$
Differentiate with respect to p_k

$$\begin{aligned} \frac{\partial \chi_g}{\partial p_k} &= \frac{\partial x_g}{\partial p_k} + \frac{\partial x_g}{\partial m}\frac{\partial e}{\partial p_k} \\ &= \frac{\partial x_g}{\partial p_k} + \frac{\partial x_g}{\partial m}\chi_k \text{ (Shephard's Lemma)} \end{aligned}$$

The Slutsky equation ($g = k$):

$$\frac{\partial x_g}{\partial p_g} = \underbrace{\frac{\partial \chi_g}{\partial p_g}}_{\substack{\leq 0 \\ \text{Hicksian} \\ \text{law of demand}}} - \underbrace{\frac{\partial x_g}{\partial m}}_{\substack{> 0 \\ \text{for normal goods}}} \chi_g .$$

The Slutsky equation for the money budget

Implications

The Slutsky equation:

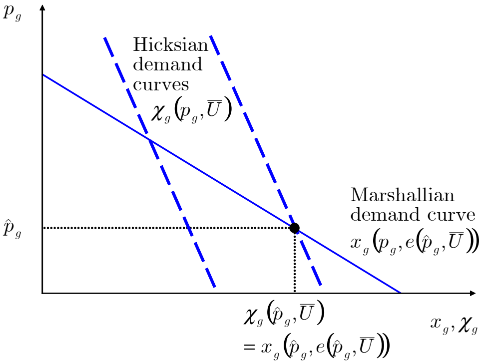
$$\frac{\partial x_g}{\partial p_g} = \underbrace{\frac{\partial \chi_g}{\partial p_g}}_{\substack{\leq 0 \\ \text{Hicksian} \\ \text{law of demand}}} - \underbrace{\frac{\partial x_g}{\partial m}}_{\substack{> 0 \\ \text{for normal goods}}} \chi_g .$$

- g normal $\Rightarrow$ g ordinary
- g normal $\Rightarrow$ effect of a price increase stronger on Marshallian demand than on Hicksian demand
- g inferior $\Rightarrow$ income effect may outweigh substitution effect —> Giffen good

The Slutsky equation for the money budget

Assume $(\hat{p}_g, \bar{U})$.

- By duality, $\chi_g(\hat{p}_g, \bar{U}) = x_g(\hat{p}_g, e(\hat{p}_g, \bar{U}))$.
- g normal $\Rightarrow$ Hicksian demand curves steeper than Marshallian demand curves



The Slutsky equation for the endowment budget

Derivation

$$
\begin{aligned}
& \frac{\partial x_g^{\text{endowment}}\left(p,\omega\right)}{\partial p_k} \\
= \quad & \frac{\partial x_g^{\text{money}}\left(p, p\cdot\omega\right)}{\partial p_k} \\
= \quad & \frac{\partial x_g^{\text{money}}}{\partial p_k} + \frac{\partial x_g^{\text{money}}}{\partial m}\frac{\partial\left(p\cdot\omega\right)}{\partial p_k} \\
= \quad & \frac{\partial x_g^{\text{money}}}{\partial p_k} + \frac{\partial x_g^{\text{money}}}{\partial m}\omega_k \text{ (definition of dot product)} \\
= \quad & \left(\frac{\partial \chi_g}{\partial p_k} - \frac{\partial x_g^{\text{money}}}{\partial m}\chi_k\right) + \frac{\partial x_g^{\text{money}}}{\partial m}\omega_k \text{ (money-budget Slutsky equati} \\
= \quad & \frac{\partial \chi_g}{\partial p_k} + \frac{\partial x_g^{\text{money}}}{\partial m}\left(\omega_k - \chi_k\right).
\end{aligned}
$$

The Slutsky equation for the endowment budget

Implications

The Slutsky equation:

$$\frac{\partial x_g^{\text{endowment}}}{\partial p_g} = \underbrace{\frac{\partial \chi_g}{\partial p_g}}_{\leq 0} + \underbrace{\frac{\partial x_g^{\text{money}}}{\partial m}}_{\substack{>0 \\ \text{for a normal} \\ \text{good } g}} \underbrace{(\omega_g - \chi_g)}_{\substack{<0 \\ \text{for net demander}}} .$$

- g normal and household net demander $\Rightarrow$ g ordinary
- g normal and household net supplier $\Rightarrow$ g may be non-ordinary

The Slutsky equation for the endowment budget

Application: consumption today versus consumption tomorrow

- The intertemporal budget equation in future value terms:

$$(1+r)\,x_1 + x_2 = (1+r)\,\omega_1 + \omega_2.$$

- The Slutsky equation:

$$\frac{\partial x_1^{\text{endowment}}}{\partial(1+r)} = \underbrace{\frac{\partial \chi_1}{\partial(1+r)}}_{\leq 0} + \underbrace{\frac{\partial x_1^{\text{money}}}{\partial m}}_{\substack{>0 \\ \text{for normal good} \\ \text{first-period consumption}}} \underbrace{(\omega_1 - \chi_1)}_{\substack{>0 \\ \text{for lender}}}.$$

The Slutsky equation for the endowment budget

Application: leisure versus consumption

- The budget equation:

$$wx_R + px_C = w24 + p\omega_C.$$

- The Slutsky equation:

$$\frac{\partial x_R^{\text{endowment}}}{\partial w} = \underbrace{\frac{\partial \chi_R}{\partial w}}_{\leq 0} + \underbrace{\frac{\partial x_R^{\text{money}}}{\partial m}}_{\substack{>0 \\ \text{for normal} \\ \text{good recreation}}} \underbrace{(24 - \chi_R)}_{\substack{\geq 0 \\ \text{by definition}}}.$$

Thus, if the wage rate increases, it may well happen that the household works ...

The Slutsky equation for the endowment budget

Application: contingent consumption

- The budget equation:

$$\frac{\gamma}{1-\gamma}x_1 + x_2 = \frac{\gamma}{1-\gamma}(A-D) + A$$

with γK – payment to the insurance if K is to be paid to the insuree in case of damage D.

- The Slutsky equation for consumption in case of damage:

$$\frac{\partial x_1^{\text{endowment}}}{\partial \frac{\gamma}{1-\gamma}} = \underbrace{\frac{\partial \chi_1}{\partial \frac{\gamma}{1-\gamma}}}_{\leq 0} + \underbrace{\frac{\partial x_1^{\text{money}}}{\partial m}}_{\substack{>0 \\ \text{for normal} \\ \text{good consumption} \\ \text{in case of damage}}} \underbrace{(A-D-\chi_1)}_{\substack{\leq 0 \\ \text{in case of} \\ \text{a nonnegative insurance}}} .$$

Compensating and equivalent variations

Overview

1. The duality approach
2. Shephard's lemma
3. The Hicksian law of demand
4. Slutsky equations
5. **Compensating and equivalent variations**

Compensating and equivalent variations

Definition

- A variation is equivalent to an event, if both (the event or the variation) lead to the same indifference curve $\longrightarrow EV\,(\text{event})$;
- A variation is compensating if it restores the individual to its old indifference curve (prior to the event) $\longrightarrow CV\,(\text{event})$.

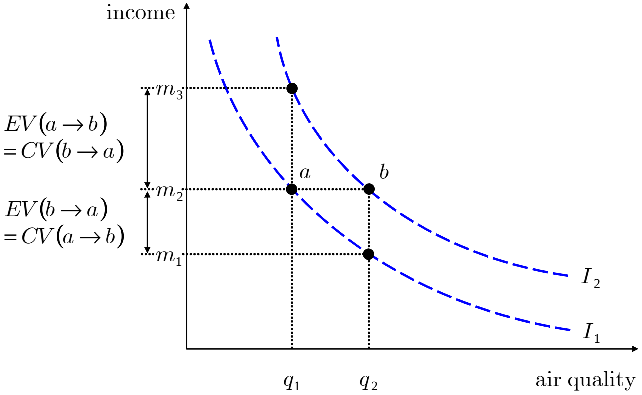
Compensating and equivalent variations

Equivalent variation	**Compensating variation**
in lieu of an event monetary variation is equivalent (i.e., achieving the same utility)	**because of an event** monetary variation compensates for event (i.e., holding utility constant)

Compensating and equivalent variations

The case of good air quality

Change of air quality:



Compensating and equivalent variations

- Compensation money $\longrightarrow$ if some amount of money is given to the individual:
  CV (degr.) – the compensation money for the degradation of the air quality.
- Willingness to pay $\longrightarrow$ if money is taken from the individual.
  EV (degr.) – the willingness to pay for the prevention of the degradation.
- If the variation turns out to be negative, exchange $-EV$ for EV or EV for $-EV$ (similarly for CV).

Compensating or equivalent variation?

Example

- Consumer's compensating variation: A consumer asks himself how much he is prepared to pay for a good.
- Consumer's equivalent variation – the compensation payment for not getting the good. You go into a shop and ask for compensation for not taking (stealing?) the good.
- Producer's compensating variation – the compensation money he gets for selling a good.
- Producer's equivalent variation: The producer asks himself how much he would be willing to pay if the good were not taken away from him.

Price changes

- The willingness to pay for the price decrease of good g:
$$CV\left(p_g^h \to p_g^l\right) = EV\left(p_g^l \to p_g^h\right).$$
- The compensation money for the price increase of good g:
$$EV\left(p_g^h \to p_g^l\right) = CV\left(p_g^l \to p_g^h\right).$$
- $CV\left(p_1^h \to p_1^l\right) < EV\left(p_1^h \to p_1^l\right)$ (for normal goods, see below);
- cv and ev – if we are not sure whether a change is good or bad.

Lemma

Consider the event of a price change from p^{old} to p^{new}. Then:

$$\begin{aligned} U^{old} &:= V\left(p^{old}, m\right) = V\left(p^{new}, m + cv\right),\ CV = |cv| \text{ and} \\ U^{new} &:= V\left(p^{new}, m\right) = V\left(p^{old}, m + ev\right), EV = |ev|. \end{aligned}$$

Price changes

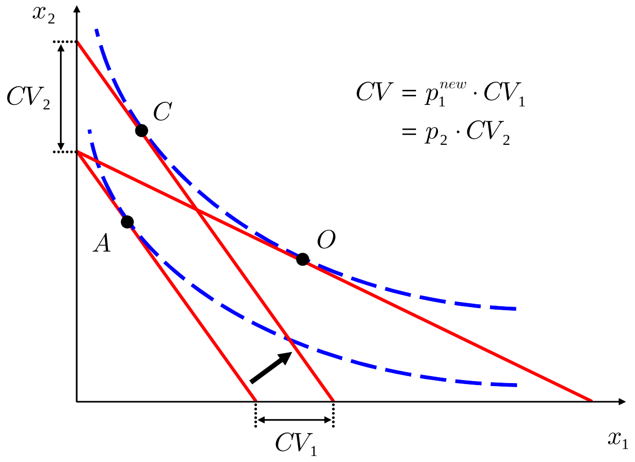
Exercise

Problem

Tell the sign of cv *and* ev *for a price increase of all goods.*

Price changes

Price increase of good 1:



Price changes

Example

Cobb-Douglas utility function: $u(x_1, x_2) = x_1^a x_2^{1-a}$ with $(0 < a < 1)$.

By a price decrease from p_1^h to $p_1^l < p_1^h$ (for example, Bahncard 50)

$$\underbrace{\left(a\frac{m}{p_1^h}\right)^a \left((1-a)\frac{m}{p_2}\right)^{1-a}}_{\text{utility at the old, high price}}$$

$$= \underbrace{\left(a\frac{m + cv\left(p_1^h \to p_1^l\right)}{p_1^l}\right)^a \left((1-a)\frac{m + cv\left(p_1^h \to p_1^l\right)}{p_2}\right)^{1-a}}_{\text{utility at the new, lower price and compensating variation}}.$$

$$cv\left(p_1^h \to p_1^l\right) = -\left(1 - \left(\frac{p_1^l}{p_1^h}\right)^a\right) m < 0.$$

Problem

Determine the equivalent variation for a price decrease in case of Cobb-Douglas utility preferences.

Problem

Determine the compensating variation and the equivalent variation for the price decrease from p_1^h to $p_1^l < p_1^h$ and the quasi-linear utility function given by

$$u(x_1, x_2) = \ln x_1 + x_2 \qquad (x_1 > 0)!$$

Assume $\frac{m}{p_2} > 1$! Hint: the household optimum is
$x(m, p) = \left(\frac{p_2}{p_1}, \frac{m}{p_2} - 1\right).$

Applying duality

Implicit definition of compensating variation

Implicit definition: $U^{old} := V\left(p^{old}, m\right) = V\left(p^{new}, m + cv\right)$
Duality equation $e\left(p, V\left(p, m\right)\right) = m$ leads to

$$\begin{aligned} e\left(p^{old}, V\left(p^{old}, m\right)\right) &= m && (1) \\ e\left(p^{new}, V\left(p^{new}, m + cv\right)\right) &= m + cv && (2) \end{aligned}$$

$\Rightarrow$

$$\begin{aligned} cv &= e\left(p^{new}, V\left(p^{new}, m + cv\right)\right) - m && (2) \\ &= e\left(p^{new}, U^{old}\right) - e\left(p^{old}, U^{old}\right) && (1) \text{ and implicit definition} \end{aligned}$$

The household is given, or is relieved of, the money necessary to uphold the old utility level.

Applying duality

Implicit definition of equivalent variation

Implicit definition: $U^{new} := V(p^{new}, m) = V\left(p^{old}, m + ev\right)$
Duality equation $e(p, V(p, m)) = m$ leads to

$$
\begin{aligned}
e\left(p^{new}, V\left(p^{new}, m\right)\right) &= m && (1) \\
e\left(p^{old}, V\left(p^{old}, m + ev\right)\right) &= m + ev && (2)
\end{aligned}
$$

$\Rightarrow$

$$
\begin{aligned}
ev &= e\left(p^{old}, V\left(p^{old}, m + ev\right)\right) - m && (2) \\
&= e\left(p^{old}, U^{new}\right) - e\left(p^{new}, U^{new}\right) && \text{(1) and implicit definitio}
\end{aligned}
$$

Assume $p^{new} < p^{old}$. The equivalent variation is the amount of money necessary to increase the household's income from $m = e\left(p^{new}, U^{new}\right)$ to $e\left(p^{old}, U^{new}\right)$.

Variations for a price change and Hicksian demand

Applying the fundamental theorem of calculus

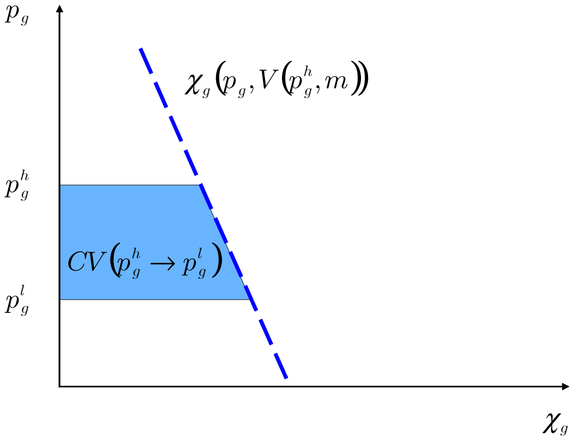
$$cv\left(p_g^h \to p_g^l\right) = -\int_{p_g^l}^{p_g^h} \chi_g\left(p_g, V\left(p_g^h, m\right)\right) dp_g$$

by (if you want)

$$\begin{aligned}
cv\left(p_g^h \to p_g^l\right) &= e\left(p_g^l, V\left(p_g^h, m\right)\right) - e\left(p_g^h, V\left(p_g^h, m\right)\right) \\
&= -\left[e\left(p_g^h, V\left(p_g^h, m\right)\right) - e\left(p_g^l, V\left(p_g^h, m\right)\right)\right] \\
&= -\, e\left(p_g, V\left(p_g^h, m\right)\right)\Big|_{p_g^l}^{p_g^h} \\
&= -\int_{p_g^l}^{p_g^h} \frac{\partial e\left(p, V\left(p_g^h, m\right)\right)}{\partial p_g} dp_g \text{ (Fundamental Theoren} \\
&= -\int_{p_g^l}^{p_g^h} \chi_g\left(p_g, V\left(p_g^h, m\right)\right) dp_g \text{ (Shephard's lemma)}
\end{aligned}$$

Variations for a price change and Hicksian demand

Applying the fundamental theorem of calculus



Theorem

Assume any good g and any price decrease from p_g^h to $p_g^l < p_g^h$.

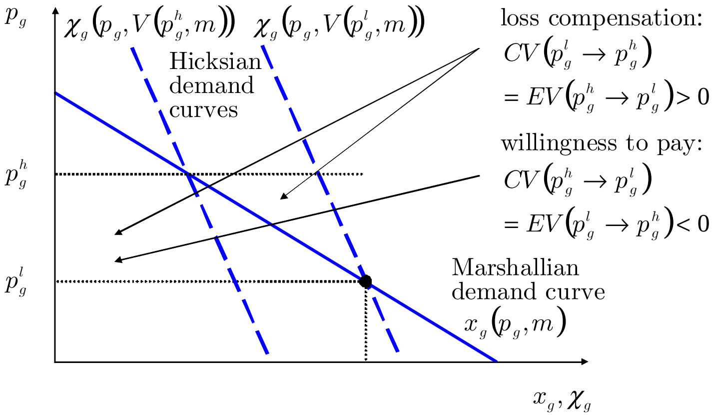
$$cv\left(p_g^h \to p_g^l\right) = -\int_{p_g^l}^{p_g^h} \chi_g\left(p_g, V\left(p_g^h, m\right)\right) dp_g \,.$$

If g is a normal good:

$$\underbrace{CV\left(p_g^h \to p_g^l\right)}_{\substack{\text{(Hicksian)} \\ \text{willingness to pay}}} \leq \underbrace{\int_{p_g^l}^{p_g^h} x_g\left(p_g\right) dp_g}_{\substack{\text{Marshallian} \\ \text{willingness to pay}}} \leq \underbrace{CV\left(p_g^l \to p_g^h\right)}_{\substack{\text{(Hicksian)} \\ \text{loss compensation}}} \,.$$

Variations for a price change and Hicksian demand

Comparisons for normal goods



Definition

The Hicksian consumer's rent at price $\hat{p}_g < p_g^{proh}$ is given by

$$\begin{aligned} CR^{Hicks}(\hat{p}_g) &:= CV\left(p_g^{proh} \to \hat{p}_g\right) \\ &= \int_{\hat{p}_g}^{p_g^{proh}} \chi_g\left(p_g, V\left(p_g^{proh}, m\right)\right) dp_g . \end{aligned}$$

Further exercises I

Problem 1
Determine the expenditure functions and the Hicksian demand function for $U(x_1, x_2) = \min(x_1, x_2)$ and $U(x_1, x_2) = 2x_1 + x_2$.
Can you confirm the duality equations

$$\begin{aligned} \chi(p, V(p, m)) &= x(p, m) \text{ and} \\ x(p, e(p, \bar{U})) &= \chi(p, \bar{U})? \end{aligned}$$

Further exercises II

Problem 2
Derive the Hicksian demand functions and the expenditure functions of the two utility functions:

(a) $U(x_1, x_2) = x_1 \cdot x_2$,
(b) $U(x_1, x_2) = \min\left(a \cdot x_1, b \cdot x_2\right)$ with $a, b > 0$.

Problem 3
Verify Roy's identity for the utility function $U(x_1, x_2) = x_1 \cdot x_2$!

Problem 4
Draw a figure that shows the equivalent variation following a price increase.