Advanced Microeconomics

The household optimum

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Part B. Household theory and theory of the firm

1. The household optimum
2. Comparative statics and duality theory
3. Production theory
4. Cost minimization and profit maximization

Nobel price 2015

In 2015, the Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel was awarded to the economist
Angus Deaton (Princeton University, NJ, USA)

for his analysis of consumption, poverty, and welfare

- By linking detailed individual choices and aggregate outcomes, his research has helped transform the fields of microeconomics, macroeconomics, and development economics.
- How do consumers distribute their spending among different goods?
- How much of society's income is spent and how much is saved?
- How do we best measure and analyze welfare and poverty?

Nobel price 2017 I

In 2017, the Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel was awarded to
Richard H. Thaler (University of Chicago, IL, USA)

for his contributions to behavioural economics

Thaler has incorporated psychologically realistic assumptions into analyses of economic decision-making. By exploring the consequences of limited rationality, social preferences, and lack of self-control, he has shown how these human traits systematically affect individual decisions as well as market outcomes.

Nobel price 2017 II

- Limited rationality:
  - Theory theory of mental accounting, explaining how people simplify financial decision-making by creating separate accounts in their minds, focusing on the narrow impact of each individual decision rather than its overall effect.
  - Aversion to losses (endowment effect) can explain why people value the same item more highly when they own it than when they don't.
- Social preferences:
  - Consumers' fairness concerns may stop firms from raising prices in periods of high demand, but not in times of rising costs.
  - Dictator game
- Lack of self-control:
  - Why are New Year's resolutions hard to keep? A planner-doer model describes the internal tension between long-term planning and short-term doing.
  - Nudging

The household optimum

Overview

1. Budget
2. The household optimum
3. Comparative statics and vocabulary
4. Applying the Lagrange method (recipe)
5. Indirect utility function
6. Consumer's rent and Marshallian demand

Definition

The expenditure for a bundle of goods $x = (x_1, x_2, ..., x_\ell)$ at a vector of prices $p = (p_1, p_2, ..., p_\ell)$ is the dot product (or the scalar product):

$$p \cdot x := \sum_{g=1}^{\ell} p_g x_g .$$

Definition

The money budget:

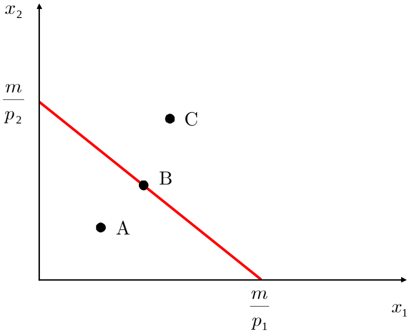
$$B(p, m) := \left\{ x \in \mathbb{R}_+^\ell : p \cdot x \leq m \right\}, p \in \mathbb{R}^\ell, m \in \mathbb{R}_+$$

The budget line:

$$\left\{ x \in \mathbb{R}_+^\ell : p \cdot x = m \right\}$$

Budget

Money budget: A two goods case



Problem

Assume that the household consumes bundle A. Identify the "left-over" in terms of good 1, in terms of good 2 and in money terms.

Problem

What happens to the budget line if

- *price p_1 doubles;*
- *if both prices double?*

Lemma

For any number $\alpha > 0$*:*

$$B(\alpha p, \alpha m) = B(p, m)$$

Problem

Fill in: For any number $\alpha > 0$*:*
$B(\alpha p, m) = B(p, ?)$.

Problem

Verify that the budget line's slope is given by $-\frac{p_1}{p_2}$ *(in case of* $p_2 \neq 0$*).*

Definition

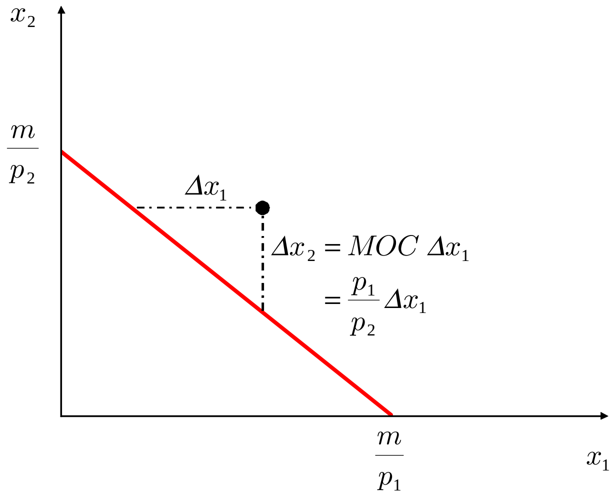
If $p_1 \geq 0$ and $p_2 > 0$,

$$MOC\left(x_1\right) = \left|\frac{dx_2}{dx_1}\right| = \frac{p_1}{p_2}$$

– the marginal opportunity cost of consuming one unit of good 1 in terms of good 2.

Budget

Marginal opportunity cost



Definition

For $p \in \mathbb{R}^\ell$ and an endowment $\omega \in \mathbb{R}^\ell_+$:

$$B(p, \omega) := \left\{ x \in \mathbb{R}^\ell_+ : p \cdot x \leq p \cdot \omega \right\}$$

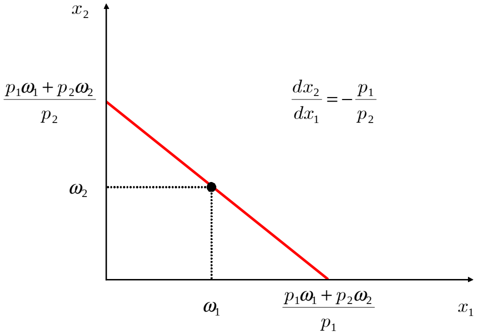
– the endowment budget.

Endowment budget

A two goods case

budget line: $p_1 x_1 + p_2 x_2 = p_1 \omega_1 + p_2 \omega_2$

marginal opportunity cost: $MOC = \left| \frac{dx_2}{dx_1} \right| = \frac{p_1}{p_2}$



Problem

What happens to the budget line if

- *price p_1 doubles;*
- *both prices double?*

Application 1
Intertemporal consumption

Notation:

- ω_1 and ω_2 – monetary income in t_1 and t_2;
- x_1 and x_2 – consumption in t_1 and t_2;
- household can borrow $(x_1 > \omega_1)$, lend $(x_1 < \omega_1)$ or consume what it earns $(x_1 = \omega_1)$;
- r – rate of interest.

Consumption in t_2:

$$x_2 = \underbrace{\omega_2}_{\text{second-period income}} + \underbrace{(\omega_1 - x_1)}_{\text{amount borrowed } (<0) \text{ or lended } (>0)} + \underbrace{r(\omega_1 - x_1)}_{\text{interest payed } (<0) \text{ or earned } (>0)}$$

$$= \omega_2 + (1 + r)(\omega_1 - x_1)$$

- *borrow* verwandt mit
  - *borgen* und
  - *bergen* („in Sicherheit bringen“) wie in *Herberge* („ein das Heer bergender Ort“)
- *lend* verwandt mit
  - *Lehen* („zur Nutzung verliehener Besitz“) und
  - *leihen*, verwandt mit
    - lateinischstämmig *Relikt* („Überrest“) und *Reliquie* („Überbleibsel oder hochverehrte Gebeine von Heiligen“) und mit
    - griechischstämmig *Eklipse* („Ausbleiben der Sonne oder des Mondes“ $>$ „Sonnen- bzw. Mondfinsternis“) und auch mit
    - griechischstämmig *Ellipse* (in der Geometrie ein Langkreis, bei dem die Höhe geringer ist als die Breite und insofern ein Mangel im Vergleich zum Kreis vorhanden ist – agr. *elleipsis* ($\acute{\varepsilon}\lambda\lambda\varepsilon\iota\psi\iota\varsigma$) bedeutet „Ausbleiben“ $>$ „Mangel“

Application 1

Intertemporal consumption

2 ways to rewrite the budget equation:

- in future-value terms:

$$(1+r)\,x_1 + x_2 = (1+r)\,\omega_1 + \omega_2,$$

- in present-value terms:

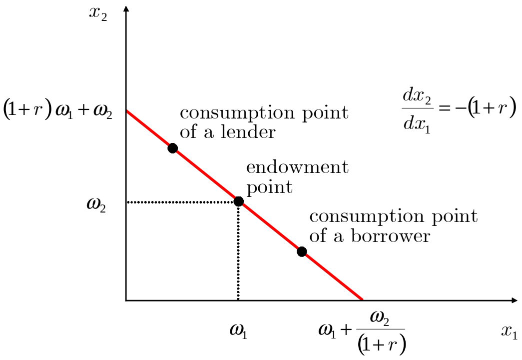
$$x_1 + \frac{x_2}{1+r} = \omega_1 + \frac{\omega_2}{1+r}.$$

Application 1

Intertemporal consumption

budget line: $(1+r)\,x_1 + x_2 = (1+r)\,\omega_1 + \omega_2$

marginal opportunity cost: $MOC = \left|\frac{dx_2}{dx_1}\right| = 1 + r$



Problem

What happens to the budget line if the interest rate decreases?

Notation:

- x_R – recreational hours ($0 \leq x_R \leq 24 = \omega_R$) $\rightarrow$ good 1;
- household works $24 - x_R$ hours;
- x_C – real consumption $\rightarrow$ good 2;
- w – the wage rate;
- ω_C – the real non-labor income;
- p – the price index.

Application 2

Leisure versus consumption

- Household's consumption in nominal terms:

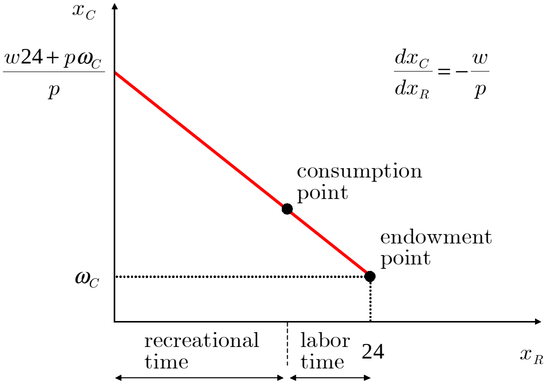
$$px_C = p\omega_C + w\left(24 - x_R\right)$$

- Household's consumption in endowment-budget form:

$$wx_R + px_C = w24 + p\omega_C$$

budget line: $wx_R + px_C = w24 + p\omega_C$

marginal opportunity cost: $MOC = \left|\frac{dx_C}{dx_R}\right| = \frac{w}{p}$



Problem

What happens to the budget line if the wage rate increases?

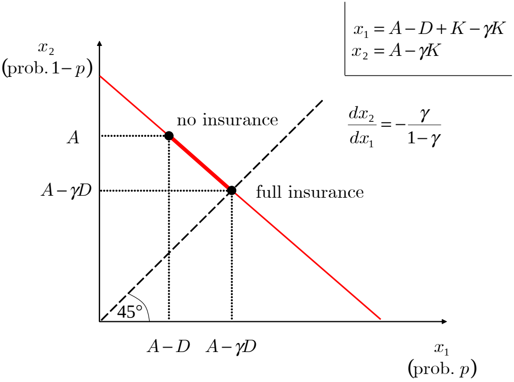
Notation:

- A – a household wealth;
- p – the probability of a bad event;
- D – possible damage;
- $L = [A - D, A; p, 1 - p]$ – lottery without insurance
- K – insurance sum;
- γ – insurance rate
- γK – insurance premium.

- $L = [x_1, x_2; p, 1 - p]$ – lottery with insurance where
  - $x_1 = A - D + K - \gamma K = A - D + (1 - \gamma) K$ $\rightarrow$ insured event
  - $x_2 = A - \gamma K$
- special cases
  - $x_1 = A - D, x_2 = A$ $\rightarrow$ no insurance ($K := 0$).
  - $x_1 = x_2 = A - \gamma D$ $\rightarrow$ full insurance ($K := D$).

budget line: $x_1 + \frac{1-\gamma}{\gamma}x_2 = (A-D) + \frac{1-\gamma}{\gamma}A$

marginal opportunity cost: $MOC = \left|\frac{dx_2}{dx_1}\right| = \frac{\gamma}{1-\gamma}$



Problem

Interpret the part of the budget line

- *right of the full-insurance point;*
- *left of the no-insurance point!*

The household optimum

1. Budget
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The household's decision problem

Definition

Best response function x^R:

$$x^R(B) \ : \ = \left\{ x \in B \colon \text{ there is no } x' \in B \text{ with } x' \succ x \right\} \text{ or}$$
$$x^R(B) \ : \ = \arg\max_{x \in B} U(x)$$

Any x^* from $x^R(B)$ – a household optimum.
Notation: also $x^R(p, m)$ or $x^R(p, \omega)$ or just $x(p)$

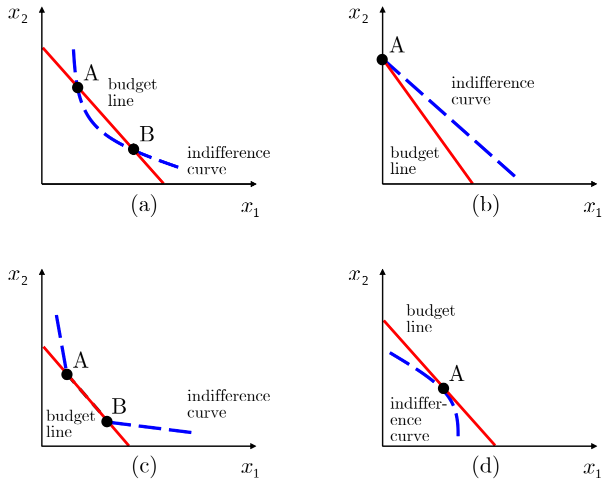
Lemma

For any number $\alpha > 0$:

$$x^R(\alpha p, \alpha m) = x^R(p, m)$$

The household's decision problem

Exercise 1



Problem

Assume monotonicity of preferences. Are the highlighted points A or B optima?

The household's decision problem

Exercise 2

Problem

Assume a household's decision problem $(B(p,\omega), \precsim)$.
$x^R(B)$ *consists of the bundles* x *that fulfill the two conditions:*

1. *The household can afford* x*:*

$$p \cdot x \leq p \cdot \omega$$

2. *There is no other bundle* y *that the household can afford and that he prefers to* x*:*

$$y \succ x \Rightarrow ??$$

Substitute the question marks by an inequality.

MRS versus MOC

Marginal willingness to pay: $MRS = \left| \frac{dx_2}{dx_1} \right|$

If the household consumes one additional unit of good 1, how many units of good 2 can he forgo so as to remain indifferent.

movement on the indifference curve

Marginal opportunity cost: $MOC = \left| \frac{dx_2}{dx_1} \right|$

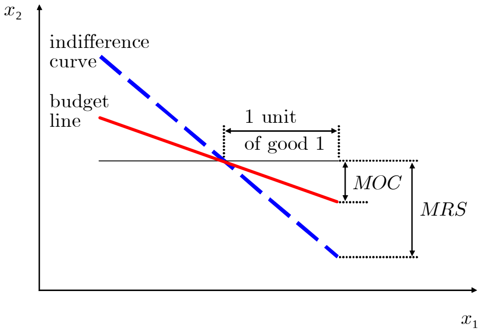
If the household consumes one additional unit of good 1, how many units of good 2 does he have to forgo so as to remain within his budget.

movement on the budget line

MRS versus MOC

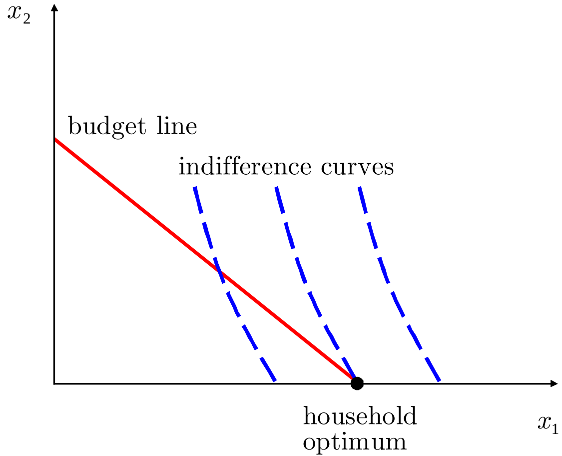
$$MRS = \underbrace{\left|\frac{dx_2}{dx_1}\right|}_{\substack{\text{absolute value} \\ \text{of the slope of} \\ \text{the indifference curve}}} > \underbrace{\left|\frac{dx_2}{dx_1}\right|}_{\substack{\text{absolute value} \\ \text{of the slope of} \\ \text{the budget line}}} = MOC$$

$\Rightarrow$ increase x_1 (if possible)



MRS versus MOC

$MRS > MOC \Rightarrow$ increase x_1 (if possible)



MRS versus MOC

Alternatively: the household tries to maximize $U\left(x_1, \frac{m}{p_2} - \frac{p_1}{p_2}x_1\right)$.

- Consume 1 additional unit of good 1
  - utility increases by $\frac{\partial U}{\partial x_1}$
  - reduction in x_2 by $MOC = \left|\frac{dx_2}{dx_1}\right| = \frac{p_1}{p_2}$ and hence utility decrease by $\frac{\partial U}{\partial x_2}\left|\frac{dx_2}{dx_1}\right|$ (chain rule)
- Thus, increase consumption of good 1 as long as

$$\underbrace{\frac{\partial U}{\partial x_1}}_{\substack{\text{marginal benefit} \\ \text{of increasing } x_1}} > \underbrace{\frac{\partial U}{\partial x_2}\left|\frac{dx_2}{dx_1}\right|}_{\substack{\text{marginal cost} \\ \text{of increasing } x_1}}$$

$$\text{or } MRS = \frac{\frac{\partial U}{\partial x_1}}{\frac{\partial U}{\partial x_2}} > \left|\frac{dx_2}{dx_1}\right| = MOC$$

Household optimum

Cobb-Douglas utility function

$U(x_1, x_2) = x_1^a x_2^{1-a}$ with $0 < a < 1$

The two optimality conditions

- $MRS = \frac{\frac{\partial U}{\partial x_1}}{\frac{\partial U}{\partial x_2}} = \frac{a}{1-a}\frac{x_2}{x_1} \overset{!}{=} \frac{p_1}{p_2}$ and
- $p_1 x_1 + p_2 x_2 \overset{!}{=} m$

yield the household optimum

$$\begin{aligned} x_1^*(m, p) &= a\frac{m}{p_1}, \\ x_2^*(m, p) &= (1-a)\frac{m}{p_2}. \end{aligned}$$

Household optimum

Perfect substitutes

$U(x_1, x_2) = ax_1 + bx_2$ with $a > 0$ and $b > 0$

An increase of good 1 enhances utility if

$$\frac{a}{b} = MRS > MOC = \frac{p_1}{p_2}$$

holds. Therefore

$$x^*(m, p) = \begin{cases} \left(\frac{m}{p_1}, 0\right), & \frac{a}{b} > \frac{p_1}{p_2} \\ \left\{\left(x_1, \frac{m}{p_2} - \frac{p_1}{p_2}x_1\right) \in \mathbb{R}_+^2 : x_1 \in \left[0, \frac{m}{p_1}\right]\right\} & \frac{a}{b} = \frac{p_1}{p_2} \\ \left(0, \frac{m}{p_2}\right) & \frac{a}{b} < \frac{p_1}{p_2} \end{cases}$$

Household optimum

Concave preferences

$$U(x_1, x_2) = x_1^2 + x_2^2$$

An increase of good 1 enhances utility if

$$\frac{x_1}{x_2} = \frac{2x_1}{2x_2} = \frac{\frac{\partial U}{\partial x_1}}{\frac{\partial U}{\partial x_2}} = MRS > MOC = \frac{p_1}{p_2}$$

holds. Therefore, corner solutions:

$$x^*(m, p) = \begin{cases} \left(\frac{m}{p_1}, 0\right), & p_1 < p_2 \\ \left\{\left(\frac{m}{p_1}, 0\right), \left(0, \frac{m}{p_2}\right)\right\} & p_1 = p_2 \\ \left(0, \frac{m}{p_2}\right) & p_1 > p_2 \end{cases}$$

Household optimum

Dixit-Stiglitz preferences for love of variety

$$U(x_1, ..., x_\ell) = \left(\sum_{j=1}^{\ell} x_j^{\frac{\varepsilon-1}{\varepsilon}}\right)^{\frac{\varepsilon}{\varepsilon-1}} \text{ with } \varepsilon > 1$$

with

$$\frac{\frac{\partial U(x_1,x_2,...,x_\ell)}{\partial x_g}}{\frac{\partial U(x_1,x_2,...,x_\ell)}{\partial x_k}} = \frac{\frac{\varepsilon}{\varepsilon-1}\left(\sum_{j=1}^{\ell} x_j^{\frac{\varepsilon-1}{\varepsilon}}\right)^{\frac{\varepsilon}{\varepsilon-1}-1}\left(\frac{\varepsilon-1}{\varepsilon} x_g^{\frac{\varepsilon-1}{\varepsilon}-1}\right)}{\frac{\varepsilon}{\varepsilon-1}\left(\sum_{j=1}^{\ell} x_j^{\frac{\varepsilon-1}{\varepsilon}}\right)^{\frac{\varepsilon}{\varepsilon-1}-1}\left(\frac{\varepsilon-1}{\varepsilon} x_k^{\frac{\varepsilon-1}{\varepsilon}-1}\right)}$$

$$= \frac{x_g^{\frac{\varepsilon-1}{\varepsilon}-1}}{x_k^{\frac{\varepsilon-1}{\varepsilon}-1}} = \frac{x_g^{\frac{-1}{\varepsilon}}}{x_k^{\frac{-1}{\varepsilon}}} = \left(\frac{x_g}{x_k}\right)^{\frac{-1}{\varepsilon}} = \left(\frac{x_k}{x_g}\right)^{\frac{1}{\varepsilon}} \stackrel{!}{=} \frac{p_g}{p_k}$$

and, in the case of two goods ($\ell = 2$), we obtain

$$x_1^*(m,p) = \frac{m}{p_1 + p_2\left(\frac{p_1}{p_2}\right)^{\varepsilon}} = \frac{m}{p_1 + p_2^{1-\varepsilon}p_1^{\varepsilon}},\ x_2^*(m,p) = \frac{m}{p_2 + p_1^{1-\varepsilon}p_2^{\varepsilon}}$$

Household optimum and monotonicity I

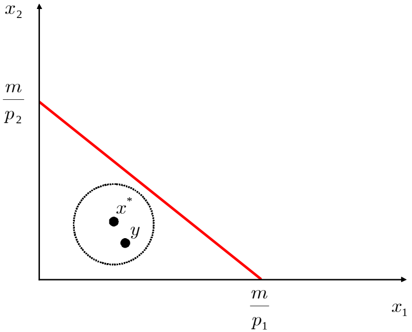
Lemma

Let $x^*(p, m)$ be a household optimum. Then

- *local nonsatiation implies $p \cdot x^* = m$ (Walras' law); and ...*

Proof.

$p \cdot x^* \leq m$ (why?).
Assume: $p \cdot x^* < m$
$\Rightarrow$ contradiction! □



Household optimum and monotonicity II

Lemma

Let $x^*(p, m)$ be a household optimum. Then

- ...
- *strict monotonicity implies* $p >> 0$;
- *local nonsatiation and weak monotonicity imply* $p \geq 0$.

Proof.

- Assume $p_g \leq 0 \Rightarrow$ household can be made better off by consuming more of good g (strict monotonicity). Contradiction!
- Assume $p_g < 0 \Rightarrow$ household can "buy" additional units of g without being worse off (weak monotonicity). Household has additional funding for preferred bundles (nonsatiation). Contradiction!

□

Comparative statics and vocabulary

Vocabulary

Definition

- The (Marshallian) demand function for good $g \longrightarrow x_g\left(p_g\right)$;
- The cross demand function for good g with respect to p_k of good $k \neq g \longrightarrow x_g\left(p_k\right)$;
- The Engel function for good $g \longrightarrow x_g\left(m\right)$.

In case of an endowment budget the household is called

- a net supplier of good g if $x_g\left(p, \omega\right) < \omega_g$ and
- a net demander if $x_g\left(p, \omega\right) > \omega_g$.

Vocabulary

Definition

A good g is:

- ordinary if
$$\frac{\partial x_g}{\partial p_g} \leq 0$$
(non-ordinary otherwise) $\longrightarrow$ slope of demand curve;
- normal if
$$\frac{\partial x_g}{\partial m} \geq 0$$
(inferior otherwise) $\longrightarrow$ slope of Engel curve;

Problem

Consider the demand function $x = a\frac{m}{p}$, $a > 0$.
Is the good an ordinary and/or a normal good?

Vocabulary

Definition

A good g is:

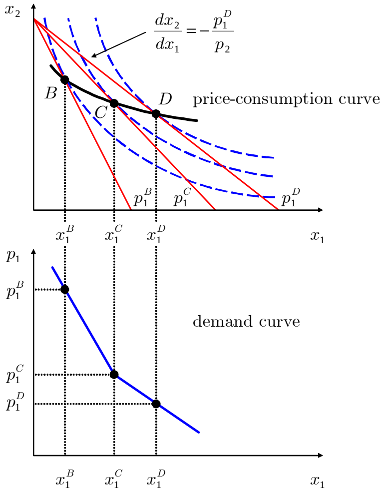
- a substitute of good k if
$$\frac{\partial x_g}{\partial p_k} \geq 0;$$
- a complement of good k if
$$\frac{\partial x_g}{\partial p_k} \leq 0.$$

Problem

Consider the demand functions $x_1 = a\frac{m}{p_1}$ *and* $x_2 = (1-a)\frac{m}{p_2}$, $0 < a < 1$, *and find out whether good* 1 *is a substitute or a complement of good* 2*!*

Price-consumption curve and demand curve

Deriving the demand curve graphically



Problem

Assuming that good 1 and good 2 are complements, sketch a price-consumption curve and the associated demand curve for good 1.

Price-consumption curve and demand curve

Deriving the demand curve analytically

Assume the utility function $U(x_1, x_2) = x_1^{\frac{1}{3}} \cdot x_2^{\frac{2}{3}}$ with household optimum

$$x_1^* = \frac{1}{3}\frac{m}{p_1}, \; x_2^* = \frac{2}{3}\frac{m}{p_2}.$$

- The demand curve for good 1 is $x_1^* = f(p_1) = \frac{1}{3}\frac{m}{p_1}$.
- $x_2^* = h(x_1) = \frac{2}{3}\frac{m}{p_2}$ is already the price-consumption curve–x_2 is a constant (boring) function x_1.
- Note: It is important that h is *not* a function of p_1.

Problem

Determine, analytically, the price-consumption curve for the the case of perfect complements, $U(x_1, x_2) = \min(x_1, 2x_2)$*! Can you also find the demand function for good* 2*? Assume* $p_1 > 0$ *and* $p_2 > 0$*!*

Vocabulary

Saturation quantity and prohibitive price

Definition

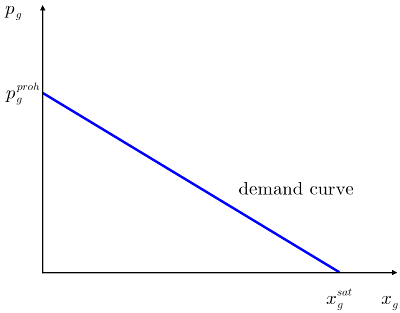
Let $x_g(p_g)$ be the quantity demanded for any $p_g \geq 0$. $\Rightarrow$

$$x_g^{sat} := x_g(0)$$

– the saturation quantity;

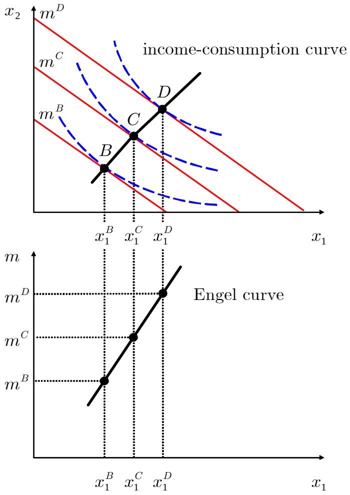
$$p_1^{proh} := \min \{p_g \geq 0 : x_g(p_g) = 0\}$$

– the prohibitive price.



Income-consumption curve and Engel curve

Deriving the Engel curve graphically



Problem

Assuming that good 1 and good 2 are complements, sketch an income-consumption curve and the associated Engel curve for good 1!

Income-consumption curve and Engel curve

Deriving the Engel curve analytically

Assume the household optimum $x_1^* = \frac{1}{3}\frac{m}{p_1}$, $x_2^* = \frac{2}{3}\frac{m}{p_2}$.

- The Engel curve for good 1 is

$$x_1^* = q\left(m\right) = \frac{1}{3}\frac{m}{p_1}.$$

- Income-consumption curve:
  - Solve good 1's demand for m and obtain $m = 3p_1x_1^*$. Subsituting in x_2^* yields $x_2^* = \frac{2}{3}\frac{m}{p_2} = \frac{2}{3}\frac{3p_1x_1^*}{p_2} = 2\frac{p_1}{p_2}x_1^*$ and hence the income-consumption curve

$$x_2^* = g\left(x_1^*\right) = 2\frac{p_1}{p_2}x_1^*.$$

- Note: It is important that g is *not* a function of m.

Problem

Determine, analytically, the income-consumption curve and the Engel-curve function for $U\left(x_1, x_2\right) = \min\left(x_1, 2x_2\right)$*!*

Defining substitutes and complements

Exercise

Problem

Determine $\frac{\partial x_1(p,m)}{\partial p_2}$ *and* $\frac{\partial x_2(p,m)}{\partial p_1}$ *for the quasi-linear utility function:*

$$U(x_1, x_2) = \ln x_1 + x_2 \qquad (x_1 > 0)!$$

Assume positive prices and $\frac{m}{p_2} > 1$, *in order to avoid a corner solution!*

Conclusion: good g can be the substitute of good k while k is not a (strict) substitute of g!

Definition

Let $x_g(p_g)$ be the demand at price p_g (other prices are held constant). $\Rightarrow$

$$\varepsilon_{x_g,p_g} := \underbrace{\frac{\frac{dx_g}{x_g}}{\frac{dp_g}{p_g}}}_{\text{mathematically doubtful}} = \frac{dx_g}{dp_g}\frac{p_g}{x_g}$$

– the price elasticity of demand.

Problem

Calculate the price elasticities of demand for the demand functions:

$$x_g(p_g) = 100 - p_g \text{ and } x_k(p_k) = \frac{1}{p_k}.$$

Vocabulary

Price elasticity of demand - application

- Drug users have inelastic demand: $|\varepsilon_{x,p}| < 1$ or $\varepsilon_{x,p} > -1$
- Then a price increase increases expenditure:

$$\begin{aligned} \frac{d\left(px\left(p\right)\right)}{dp} &= x + p\frac{dx}{dp} \\ &= x\left(1 + \frac{p}{x}\frac{dx}{dp}\right) = x\left(1 + \varepsilon_{x,p}\right) = x\left(1 - |\varepsilon_{x,p}|\right) > 0. \end{aligned}$$

- Political implication:
  Making drugs expensive
  - by taxing them or
  - by criminalizing selling or buying

  increases expenditure and hence drug-related crime (stealing money in order to finance the addiction).

Definition

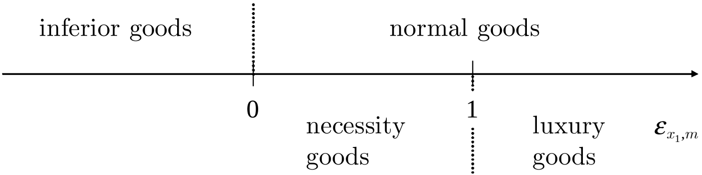
Let $x_g(m)$ be the demand at income m. The income elasticity (of demand) is denoted by $\varepsilon_{x_g,m}$ and given by

$$\varepsilon_{x_g,m} := \frac{\frac{dx_g}{x_g}}{\frac{dm}{m}} = \frac{dx_g}{dm}\frac{m}{x_g}.$$

Definition

We call a good g

- a luxury good (such as caviar) if $\varepsilon_{x_g,m} \geq 1$ holds;
- a necessity good (such as oat groats) if $0 \leq \varepsilon_{x_g,m} \leq 1$ holds.



Problem

Calculate the income elasticity of demand for the Cobb-Douglas utility function $U(x_1, x_2) = x_1^{\frac{1}{3}} \cdot x_2^{\frac{2}{3}}$*! How do you classify (demand for) good* 1*?*

Lemma

Assume local nonsatiation and the household optimum x^*. Then the average income elasticity is 1*:*

$$\sum_{g=1}^{\ell} s_g \varepsilon_{x_g,m} = 1$$

where the weights are the relative expenditures, $s_g := \frac{p_g x_g}{m}$.

- According to Walras' law, the household chooses x^* on the budget line, $p \cdot x^*(m) = m$. (How about $\ell = 1$?)
- Find the derivative of the budget equation $m = \sum_{g=1}^{\ell} p_g x_g^*(m)$ with respect to m to obtain

$$1 = \sum_{g=1}^{\ell} p_g \frac{dx_g^*}{dm}$$

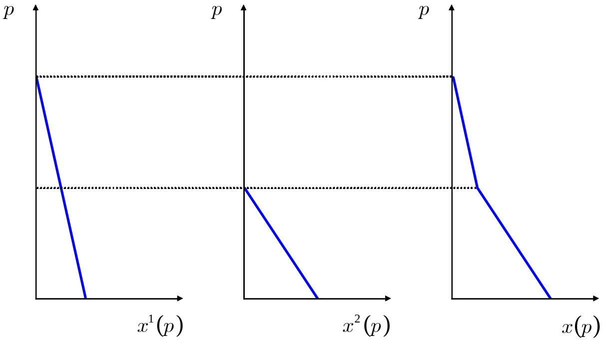
and,

- by multiplying the summands with $\frac{x_g^*}{m}\frac{m}{x_g^*} = 1$,

$$1 = \sum_{g=1}^{\ell} p_g \frac{dx_g^*}{dm} \frac{x_g^*}{m} \frac{m}{x_g^*} = \sum_{g=1}^{\ell} \frac{p_g x_g^*}{m} \frac{dx_g^*}{dm} \frac{m}{x_g^*} = \sum_{g=1}^{\ell} s_g \varepsilon_{x_g,m}$$

Definition

Let $x^i(p)$ be the demand functions of individuals $i = 1, ..., n$. $\Rightarrow$
$x(p) := \sum_{i=1}^{n} x^i(p)$ – aggregate demand.



Problem

$$\begin{aligned} x_g^1(p_g) &= \max(0, 100 - p_g), \\ x_g^2(p_g) &= \max(0, 50 - 2p_g) \text{ and} \\ x_g^3(p_g) &= \max(0, 60 - 3p_g). \end{aligned}$$

Find the aggregate demand function!
Hint: Find the prohibitive prices first!

Vocabulary

Inverse demand function

Definition

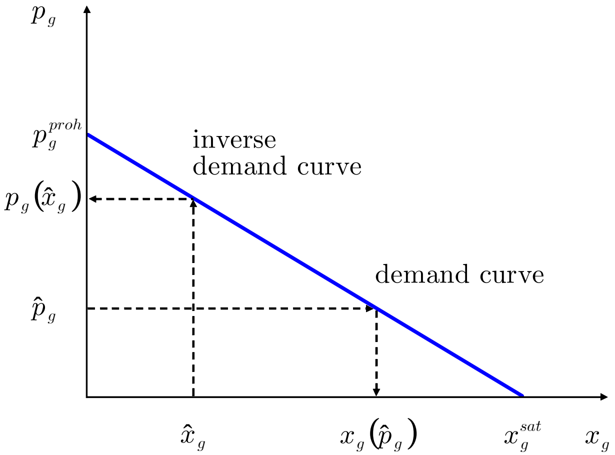
Let $x_1 : \left[0, p_1^{proh}\right] \rightarrow \left[0, x_1^{sat}\right]$ be an injective demand function. $\Rightarrow$

$$\begin{aligned} p_1 &= x_1^{-1} : \left[0, x_1^{sat}\right] \rightarrow \left[0, p_1^{proh}\right] \\ & \quad x_1 \mapsto p_1\left(x_1\right) \text{ where } p_1\left(x_1\right) \text{ is the unique price resulting in } x_1. \end{aligned}$$

– the inverse demand function.

Vocabulary

Inverse demand function



The Lagrange method

1. Budget
2. The household optimum
3. Comparative statics and vocabulary
4. **Applying the Lagrange method (recipe)**
5. Indirect utility function
6. Consumer's rent and Marshallian demand

Applying the Lagrange method (recipe)

$$L(x,\lambda) = U(x) + \lambda \left[m - \sum_{g=1}^{\ell} p_g x_g \right]$$

with:

- U – strictly quasi-concave and strictly monotonic utility function;
- prices $p >> 0$;
- $\lambda > 0$ – Lagrange multiplier - translates a budget surplus $m - \sum_{g=1}^{\ell} p_g x_g > 0$ into utility
- Increasing consumption has
  - a positive effect via U, but
  - a negative effect via decreasing budget and λ

Applying the Lagrange method (recipe)

Differentiate L with respect to x_g :

$$\frac{\partial L\left(x_1, x_2, ..., \lambda\right)}{\partial x_g} = \frac{\partial U\left(x_1, x_2, ..., x_\ell\right)}{\partial x_g} - \lambda p_g \overset{!}{=} 0 \text{ or } \frac{\partial U\left(x_1, x_2, ..., x_\ell\right)}{\partial x_g} \overset{!}{=} \lambda p_g$$

and hence, for two goods g and k

$$\frac{\frac{\partial U(x_1, x_2, ..., x_\ell)}{\partial x_g}}{\frac{\partial U(x_1, x_2, ..., x_\ell)}{\partial x_k}} \overset{!}{=} \frac{p_g}{p_k} \text{ or } MRS \overset{!}{=} MOC$$

Applying the Lagrange method (recipe)

Problem

Set the derivative of L with respect to λ equal to 0. What do you find?

λ – the shadow price of the restriction:

$$\lambda = \frac{dU}{dm}.$$

But: U does not have m as an argument so that $\frac{dU}{dm}$ is not quite correct.

Indirect utility function

1. Budget
2. The household optimum
3. Comparative statics and vocabulary
4. Applying the Lagrange method (recipe)
5. **Indirect utility function**
6. Consumer's rent and Marshallian demand

Definition

Consider a household with utility function U. $\Rightarrow$

$$\begin{aligned} V &: \mathbb{R}^{\ell} \times \mathbb{R}_{+} \to \mathbb{R}, \\ & (p, m) \mapsto V(p, m) := U(x(p, m)) \end{aligned}$$

– indirect utility function.

Utility function versus indirect utility function

function	arguments	optimal bundles
utility function	$x \in \mathbb{R}^\ell_+$	$x(p, m)$
indirect utility function	m and $p \in \mathbb{R}^\ell$	$x(p, m)$

Problem

Determine the indirect utility function for the Cobb-Douglas utility function $U(x_1, x_2) = x_1^a x_2^{1-a}$ $(0 < a < 1)$*!*

Revisiting the Lagrange multiplier

Aim: $\lambda = \frac{dU}{dm} \rightarrow \lambda = \frac{dV}{dm}$

- Differentiating the budget function with respect to m yields $\sum_{g=1}^{\ell} p_g \frac{\partial x_g}{\partial m} = \frac{dm}{dm} = 1$ (as we know from the the proof about the average income elasticity)
- Differentiating the indirect utility function $V(p, m) = U(x(p, m))$ with respect to m leads to

$$\frac{\partial V}{\partial m} = \sum_{g=1}^{\ell} \frac{\partial U}{\partial x_g} \frac{\partial x_g}{\partial m} = \sum_{g=1}^{\ell} (\lambda p_g) \frac{\partial x_g}{\partial m} = \lambda \sum_{g=1}^{\ell} p_g \frac{\partial x_g}{\partial m} = \lambda$$

Revisiting the Lagrange multiplier

The optimization condition can be rewritten as:

$$\underbrace{\frac{\partial U\left(x_1, x_2, ..., x_\ell\right)}{\partial x_g}}_{\text{marginal utility}} \overset{!}{=} \lambda p_g = \underbrace{\frac{\partial V}{\partial m} p_g}_{\text{marginal cost}}$$

Interpreting the marginal cost of consuming one additional unit of good g :

- You consume one additional unit of good g,
- your expenditure increases by p_g so that
- the income left for other goods decreases by p_g and hence
- utility decreases by $\frac{\partial V}{\partial m} p_g$.

Consumer's rent and Marshallian demand

1. Budget
2. The household optimum
3. Comparative statics and vocabulary
4. Applying the Lagrange method (recipe)
5. Indirect utility function
6. **Consumer's rent and Marshallian demand**

Marginal willingness to pay

Assume:

- x_2 – "all the other goods" (money);
- $p_2 = 1$.

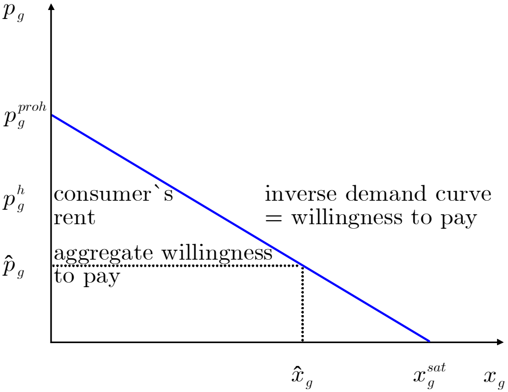
$\Rightarrow$

- Marginal willingness to pay for one extra unit of x_1 is

$$MRS = \frac{p_1}{p_2} = p_1.$$

- Inverse demand function measures (cum grano salis) the marginal willingness to pay for one extra unit of a good.

Marginal willingness to pay



Problem

$p(q) = 20 - 4q$, $p = 4$

aggregate willingness to pay? consumer's rent?

The Marshallian willingness to pay and the Marshallian consumer's rent

Definition

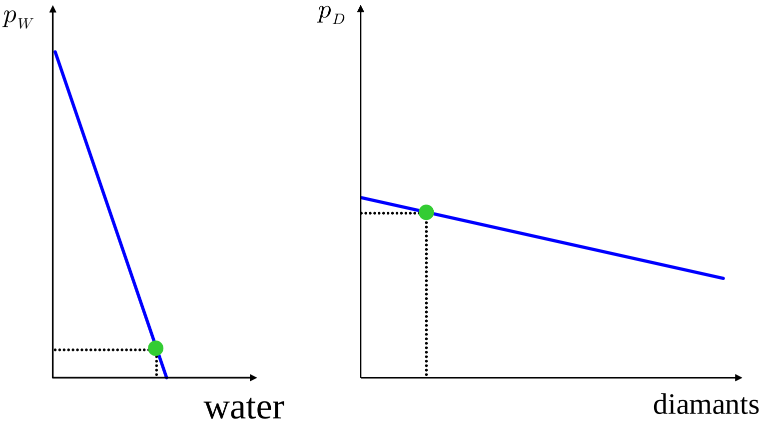
Let p_g be an inverse demand function.
The Marshallian willingness to pay

- for the quantity $\hat{x}_g$: $\int_0^{\hat{x}_g} p_g(x_g)\, dx_g$
- for a price decrease from p_g^h to $\hat{p}_g < p_g^h$: $\int_{\hat{p}_g}^{p_g^h} x_g(p_g)\, dp_g$
- for a price decrease from p_g^{proh} to $\hat{p}_g < p_g^{proh}$ (consumer's rent)

$$
\begin{aligned}
CR^{Marshall}(\hat{p}_g) &= \int_0^{x_g(\hat{p}_g)} p_g(x_g)\, dx_g - \hat{p}_g x_g(\hat{p}_g) \\
&= \int_0^{x_g(\hat{p}_g)} (p_g(x_g) - \hat{p}_g)\, dx_g = \int_{\hat{p}_g}^{p_g^{proh}} x_g(p_g)\, dp_g
\end{aligned}
$$

The diamond-water paradox

Why are diamonds more expensive than water
although water is more „useful"?



The Marshallian willingness to pay

An imperfect concept

Marshallian willingness to pay for the quantity $\hat{x}_g$
—> "sum" the prices for all the units from 0 up to $\hat{x}_g$
But:

- According to Marshallian demand, the consumers pay one price for all the units.
- In contrast, the integral $\int_0^{\hat{x}_g} p_g(x_g)\, dx_g$ presupposes that the consumer pays (in general) different prices for the first, the second, and so on, units.

Now, if higher prices are to be paid for the first units, the household has less income available to spend on the additional units:

- demand is lowered
- consumer's willingness to pay or consumer's rent is lower
- Marshallian willingness to pay exaggerates "real" consumers' rent —> next chapter

Nobel price 2017 III

- Mental accounting: A couple saves for loong-term goal (buying house) with interest rate r_s and takes out a loan for middle-term acquisition (car) with interest rate $r_h > r_s$.
- Loss aversion:
  - mug given: do you want to exchange against pen?
  - pen given: do you want to exchange against mug?

  Many people say "no" to both questions.
  Thus, endowments influence preferences (consider two endowments with the same value).
- Fairness: A friend offers to buy a beer (Budweiser) at your cost in
  - a fancy resort hotel
  - a shabby grocery store

  He asks your willingness to pay. Is it larger for the first seller than for the second?

Further exercises

Problem 1
Discuss the units in which to measure price, quantity, expenditure. If you are right, expenditure should be measured in the same units as the product of price and quantity.

Problem 2
Sketch budget lines:

- Time $T = 18$ and money $m = 50$ for football F (good 1) or basket ball B (good 2) with prices
  - $p_F = 5$, $p_B = 10$ in monetary terms,
  - $t_F = 3$, $t_B = 2$ and temporary terms
- Two goods, bread (good 1) and other goods (good 2). Transfer in kind with and without probhibition to sell:
  - $m = 300$, $p_B = 2$, $p_{\text{other}} = 1$
  - Transfer in kind: $B = 50$

Further exercises

Problem 3

Assume two goods 1 and 2. Abba faces a price p for good 1 in terms of good 2. Think of good 2 as the numeraire good with price 1. Abba's utility functions U is given by $U(x_1, x_2) = \sqrt{x_1} + x_2$. His endowment is $\omega = (25, 0)$. Find Abba's optimal bundle. *Hint: Distinguish* $p \geq \frac{1}{10}$ *and* $p < \frac{1}{10}$*!*

Problem 4

Show by way of example that $\frac{\partial x_1(p,m)}{\partial p_1} < 0$ and $\frac{\partial x_1(p,\omega)}{\partial p_1} > 0$ may well happen. *Hint: use perfect complements.*

Further exercises

Problem 5

Derive the indirect utility functions of the following utility functions:

(a) $U(x_1, x_2) = x_1 \cdot x_2$,

(b) $U(x_1, x_2) = \min\{a \cdot x_1, b \cdot x_2\}$ where $a, b > 0$ holds,

(c) $U(x_1, x_2) = a \cdot x_1 + b \cdot x_2$ where $a, b > 0$ holds.

Problem 6

Consider the preferences given by the utility function $U(x_1, x_2) = x_1 + 2x_2$. Find $x(p, m)$. Sketch the demand function for good 1. Sketch the Engel curve for good 1 for the case of $p_1 < \frac{1}{2}p_2$ while observing the usual convention that the x_1-axis is the abscissa!