Advanced Microeconomics

Decisions under risk

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Part A. Basic decision and preference theory

1. Decisions in strategic (static) form
2. Decisions in extensive (dynamic) form
3. Ordinal preference theory
4. **Decisions under risk**

Decisions under risk

Overview

1. Simple and compound lotteries
2. The St. Petersburg lottery
3. Preference axioms for lotteries and von Neumann Morgenstern utility
4. Risk attitudes

Simple and compound lotteries

How lotteries arise

Lotteries may arise from decision situations such as

		state of the world	
		bad weather, $\frac{1}{4}$	good weather, $\frac{3}{4}$
strategy	production of umbrellas	100	81
	production of sunshades	64	121

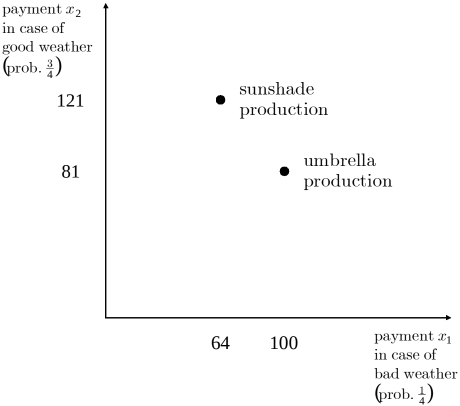
They can be understood as

- bundles of goods;
- extensive-form decision situations;
- "payoffs"

Simple lotteries as bundles and trees

Lotteries as bundles of goods

$$L_{\text{umbrella}} = \left[100, 81; \frac{1}{4}, \frac{3}{4}\right] \text{ and } L_{\text{sunshade}} = \left[64, 121; \frac{1}{4}, \frac{3}{4}\right]$$

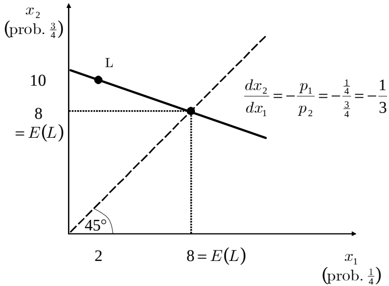


Simple lotteries as bundles and trees

Expected value of a simple lottery

Definition

$$E\left(L\right) = \sum_{j=1}^{\ell} p_j x_j,\, L = \left[x_1, ..., x_\ell; p_1, ..., p_\ell\right].$$



$$\begin{aligned} L &= \left[2, 10; \frac{1}{4}, \frac{3}{4}\right] \\ E\left(L\right) &= p_1 x_1 + p_2 x_2 \\ \Leftrightarrow \quad & x_2 = \frac{E\left(L\right)}{p_2} - \frac{p_1}{p_2} x_1 \end{aligned}$$

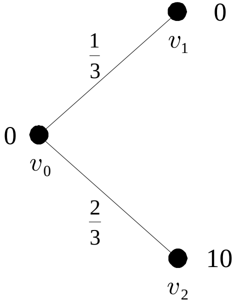
$$E\left(L\right) \underbrace{=}_{45^\circ\text{-line}} p_1 x_1 + p_2 x_1 = x_1$$

Simple lotteries as bundles and trees

Lottery as a decision situation in extensive form

Lottery $L = \left[0, 10; \frac{1}{3}, \frac{2}{3}\right]$ can be seen as a "decision" situation in extensive form

- without a decision maker,
- nature moves



Are you risk averse?

Use introspection!

Problem

Do you prefer $L_1 = \left[0, 10; \frac{1}{3}, \frac{2}{3}\right]$ *to* $L_2 = \left[5, 10; \frac{1}{4}, \frac{3}{4}\right]$*?*

Problem

Do you prefer $L = \left[95, 105; \frac{1}{2}, \frac{1}{2}\right]$ *to a certain payoff of* 100?

Compound lotteries

Lotteries as "payoffs"

Definition

Let $L_1, ..., L_\ell$ be simple lotteries. $\Rightarrow$
$\left[L_1, ..., L_\ell; p_1, ..., p_\ell\right]$ – a compound or two-stage lottery
where ℓ can be infinite.

Problem

Consider $L_1 = \left[0, 10; \frac{1}{3}, \frac{2}{3}\right]$ *and* $L_2 = \left[5, 10; \frac{1}{4}, \frac{3}{4}\right]$. *Express the compound lottery* $L = \left[L_1, L_2; \frac{1}{2}, \frac{1}{2}\right]$ *as a simple lottery! Can you draw the appropriate trees, one of length* 2 *and one of length* 1*?*

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The St. Petersburg lottery

Definition

- Imagine Peter throwing a fair coin j times until "head" occurs for the first time.
- Head (H) rather than tail (T) occurs
  - at the first coin toss (sequence H) with probability $\frac{1}{2}$,
  - at the second coin toss (sequence TH) with probability $\frac{1}{4}$ and
  - at the jth toss (sequence T...TH) with probability $\frac{1}{2^j}$.
- Peter pays 2^j to Paul if "head" occurs for the first time at the jth toss.
- St. Petersburg lottery:

$$L = \left[2, 4, 8, ..., 2^j, ...; \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, ..., \frac{1}{2^j}, ...\right].$$

- The probabilities are positive. However, do they sum up to 1?

The St. Petersburg lottery

Infinite geometric series

Fact

Infinite geometric series $\sum_{j=0}^{\infty} cq^j = c + cq + cq^2 + \ldots$ *with* $|q| < 1$ *converges:*

$$\frac{\textit{first term}}{1-\text{factor}} = \frac{c}{1-q}.$$

- The sum of the probabilities
  - $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots + \frac{1}{2^j} + \ldots = \sum_{j=1}^{\infty} \frac{1}{2^j}$
  - is an infinite geometric series
  - with $q = \frac{1}{2}$
  - so that we obtain
- $\frac{\text{first term}}{1-\text{factor}} = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$ (!)

The St. Petersburg lottery

Use introspection!

- How much are you prepared to pay for the St. Petersburg lottery?
- But
$$E\left(L\right)=\frac{1}{2}\cdot 2+\frac{1}{4}\cdot 4+\frac{1}{8}\cdot 8+...=\infty$$
- a paradox?

How to solve the paradox

- Limited resources?
- Expected utility?

Definition

$$E_u(L) = \sum_{j=1}^{\ell} p_j u(x_j)$$

– the expected utility of a simple lottery $L = [x_1, \ldots, x_\ell; p_1, \ldots, p_\ell]$ with $u : \mathbb{R} \to \mathbb{R}$. u is called a von Neumann Morgenstern utility function.

- Bounded vNM utility u?

See manuscript!

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Preference axioms

- **Completeness axiom:** Assume L_1, L_2. $\Rightarrow$

$$L_1 \succsim L_2 \text{ or } L_2 \succsim L_1$$

- **Transitivity axiom:** Assume $L_1 \succsim L_2$ and $L_2 \succsim L_3$. $\Rightarrow$

$$L_1 \succsim L_3$$

- **Continuity axiom:** Assume $L_1 \succsim L_2 \succsim L_3$. $\Rightarrow$ There is a $p \in [0,1]$ such that

$$L_2 \sim [L_1, L_3; p, 1-p]$$

- **Independence axiom:** Assume L_1, L_2, L_3 and $p > 0$. $\Rightarrow$

$$[L_1, L_3; p, 1-p] \precsim [L_2, L_3; p, 1-p] \Leftrightarrow L_1 \precsim L_2.$$

Preference axioms

Is the continuity axiom plausible?

Assume:

- L_1 – payoff of 10 €;
- L_2 – payoff of 0 €;
- L_3 – certain death.

$$L_1 \succ L_2 \succ L_3$$

Determine your p so that:

$$L_2 \sim [L_1, L_3; p, 1-p]$$

$$p = 1 \Rightarrow [L_1, L_3; 1, 0] = L_1 \succ L_2.$$

Preference axioms

Independence axiom: Exercise

Problem

Assume a decision maker who is indifferent between

$$L_1 = \left[0, 100; \frac{1}{2}, \frac{1}{2}\right] \text{ and } L_2 = \left[16, 25; \frac{1}{4}, \frac{3}{4}\right].$$

Show the indifference between

$$L_3 = \left[0, 50, 100; \frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right] \text{ and } L_4 = \left[16, 25, 50; \frac{1}{8}, \frac{3}{8}, \frac{1}{2}\right]$$

by verifying:

$$L_3 = \left[L_1, 50; \frac{1}{2}, \frac{1}{2}\right] \text{ and } L_4 = \left[L_2, 50; \frac{1}{2}, \frac{1}{2}\right].$$

Preference axioms

Independence axiom: critics

Consider the lotteries

$$L_1 = \left[12 \cdot 10^6, 0; \tfrac{10}{100}, \tfrac{90}{100}\right] \qquad L_3 = \left[1 \cdot 10^6; 1\right]$$
$$L_2 = \left[1 \cdot 10^6, 0; \tfrac{11}{100}, \tfrac{89}{100}\right] \qquad L_4 = \left[12 \cdot 10^6, 1 \cdot 10^6, 0; \tfrac{10}{100}, \tfrac{89}{100}, \tfrac{1}{100}\right]$$

- Do you prefer L_1 to L_2 and/or L_3 to L_4?
- Many people prefer L_1 to L_2 and L_3 to L_4.
- But

$$\begin{aligned}
L_1 \succ L_2 &\Rightarrow \left[L_1, L_3; \frac{1}{2}, \frac{1}{2}\right] \succ \left[L_2, L_3; \frac{1}{2}, \frac{1}{2}\right] \text{ (independence)}\\
L_3 \succ L_4 &\Rightarrow \left[L_2, L_3; \frac{1}{2}, \frac{1}{2}\right] \succ \left[L_2, L_4; \frac{1}{2}, \frac{1}{2}\right] \text{ (independence)}\\
&\Rightarrow \left[L_1, L_3; \frac{1}{2}, \frac{1}{2}\right] \succ \left[L_2, L_4; \frac{1}{2}, \frac{1}{2}\right] \text{ (transitivity)}
\end{aligned}$$

yields a contradiction! —> see next slide

Preference axioms

Exercise

Problem

Reduce $\left[L_1, L_3; \frac{1}{2}, \frac{1}{2}\right]$ *and* $\left[L_2, L_4; \frac{1}{2}, \frac{1}{2}\right]$ *to simple lotteries!*

A utility function for lotteries

vNM utility function

Theorem

Preferences between lotteries obey the four axioms iff there is $u : \mathbb{R}_+ \to \mathbb{R}$ *such that*

$$L_1 \succsim L_2 \Leftrightarrow E_u\left(L_1\right) \geq E_u\left(L_2\right)$$

holds for all $L_1, L_2 \in \mathcal{L}$.

- u *represents* $\succsim$ *on* $\mathcal{L}$;
- u *– vNM utility function.*

Distinguish between:

- $u : \mathbb{R}_+ \to \mathbb{R}$ – vNM utility function (domain: payoffs);
- $E_u : \mathcal{L} \to \mathbb{R}$ – expected utility (domain: lotteries).

A utility function for lotteries

Transformations

Definitions

u vNM utility function. v is called an affine transformation of u if v obeys $v(x) = a + bu(x)$ for $a \in \mathbb{R}$ and $b > 0$.

Lemma

If u represents the preferences $\succsim$, so does any utility function v that is an affine transformation of u.

Problem

Find a vNM utility function that is simpler than $u(x) = 100 + 3x + 9x^2$ while representing the same preferences.

A utility function for lotteries

Exercise

Problem

Consider:

$$L^A := \left[x_1^A, ..., x_{\ell_A}^A; p_1^A, ..., p_{\ell_A}^A\right] \text{ and } L^B := \left[x_1^B, ..., x_{\ell_B}^B; p_1^B, ..., p_{\ell_B}^B\right].$$

Let v be an affine transformation of u.
Show:

$$E_u\left(L^A\right) \geq E_u\left(L^B\right) \Leftrightarrow E_v\left(L^A\right) \geq E_v\left(L^B\right).$$

The construction of the vNM utility function

Consider:

- L_{bad} and L_{good} ($L_{good} \succ L_{bad}$);
- L so that $L_{good} \succsim L \succsim L_{bad}$.

$\Rightarrow$ By the continuity axiom, there exists $p(L)$ such that

$$L \sim [L_{good}, L_{bad}; p(L), 1 - p(L)]$$

Problem

Find $p(L_{good})$ and $p(L_{bad})$! Hint: Translate $L \sim [L_{good}, L_{bad}; p(L), 1 - p(L)]$ into a statement on expected utilities.

The construction of the vNM utility function

$L := [x; 1] \Rightarrow$

$$u(x) := p(L)$$

– a vNM utility function.

- The decision maker is indifferent between x and $[L_{good}, L_{bad}; u(x), 1 - u(x)]$.
- $u(x)$ is a value between 0 (the probability for L_{bad}) and 1 (the probability for L_{good})
- u represents the preferences of the decision maker (as shown by Myerson, 1991, pp. 12).

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Concave and convex functions

Definition

Given: $f : M \rightarrow \mathbb{R}$ (function on a convex domain $M \subseteq \mathbb{R}$). $\Rightarrow$

- f is concave if

$$f\left(kx + (1-k)\,y\right) \geq kf\left(x\right) + (1-k)\,f\left(y\right)$$

for all $x, y \in M$ and for all $k \in [0,1]$ (with $\leq$ – convex).

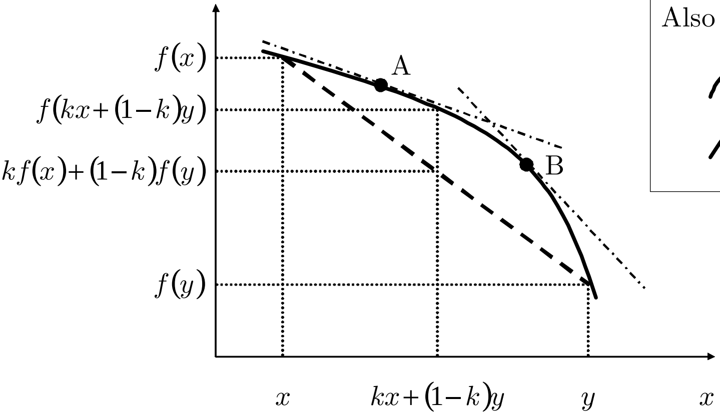
- f is strictly concave if

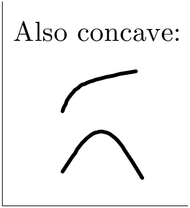
$$f\left(kx + (1-k)\,y\right) > kf\left(x\right) + (1-k)\,f\left(y\right)$$

holds for all $x, y \in M$ with $x \neq y$ and for all $k \in (0,1)$ (with $<$ – strictly convex).

Concave and convex functions

Concavity

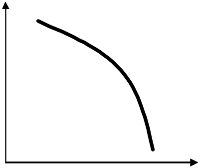
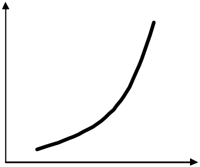
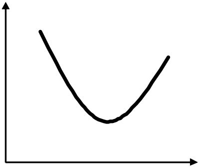




The line connecting $f(x)$ and $f(y)$ lies below the graph.

Concave and convex functions

… and quasi-concavity

	quasi-concave	not quasi-concave
concave		concavity implies quasi-concavity
not concave		

Concave and convex functions

The second derivative

Lemma

Let $f : M \to \mathbb{R}$ with convex domain $M \subseteq \mathbb{R}$ be twice differentiable.

- *f is concave on $M \subseteq \mathbb{R}$ iff*

$$f''(x) \leq 0$$

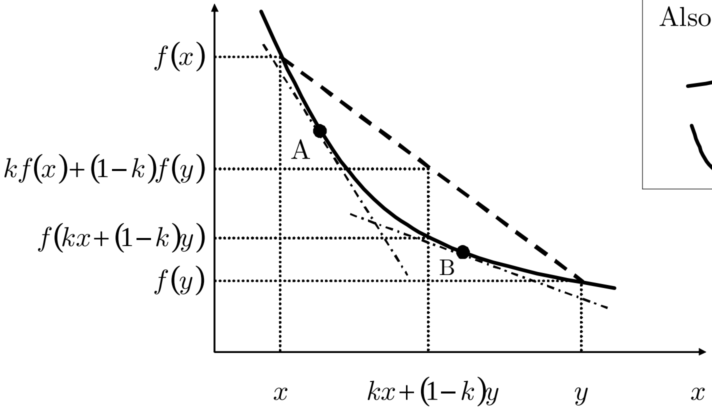
  holds for all $x \in M$.
- *f is convex on $M \subseteq \mathbb{R}$ iff*

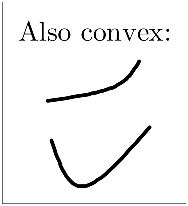
$$f''(x) \geq 0$$

  holds for all $x \in M$.

Concave and convex functions

Convexity





The line connecting $f(x)$ and $f(y)$ lies above the graph.

Concave and convex functions

Convexity: Exercise

Problem

Comment: If a function $f : \mathbb{R} \to \mathbb{R}$ *is not concave, it is convex.*

Risk aversion and risk loving

Definition

Assume $\succsim$ on $\mathcal{L}$. A decision maker is:

- risk neutral if

$$L \sim [E(L); 1] \text{ or } E_u(L) = u(E(L));$$

- risk-averse if

$$L \precsim [E(L); 1] \text{ or } E_u(L) \leq u(E(L));$$

- risk-loving if

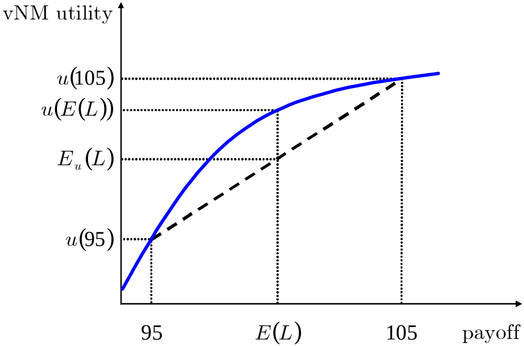
$$L \succsim [E(L); 1] \text{ or } E_u(L) \geq u(E(L))$$

for all lotteries $L \in \mathcal{L}$.

Risk aversion and risk loving

Risk aversion

$$L = \left[95, 105; \tfrac{1}{2}, \tfrac{1}{2}\right]$$
$$E_L = 100$$
$$u(100) = u(E(L)) > E_u(L) = \tfrac{1}{2}u(95) + \tfrac{1}{2}u(105)$$



Risk aversion and risk loving

Lemma

Assume $\succsim$ on $\mathcal{L}$ and an associated vNM utility function u.
A decision maker is:

- *risk neutral iff u is an affine function (i.e., $u(x) = ax + b$, $a > 0$);*
- *risk-averse iff u is concave;*
- *risk-loving iff u is convex.*

Risk aversion and risk loving

Exercise

Problem

Do the preferences characterized by the following utility functions exhibit risk-averseness?

- $u_1(x) = x^2$, $x > 0$
- $u_2(x) = 2x + 3$
- $u_3(x) = ln(x)$, $x > 0$
- $u_4(x) = -e^{-x}$
- $u_5(x) = \frac{x^{1-\theta}}{1-\theta}, \theta > 0, \theta \neq 1$

Certainty equivalent and risk premium

Definition

For any $L \in \mathcal{L}$, the payoff $CE\left(L\right)$ is the certainty equivalent of L, if

$$L \sim \left[CE\left(L\right); 1\right]$$

holds.

Definition

For any $L \in \mathcal{L}$:

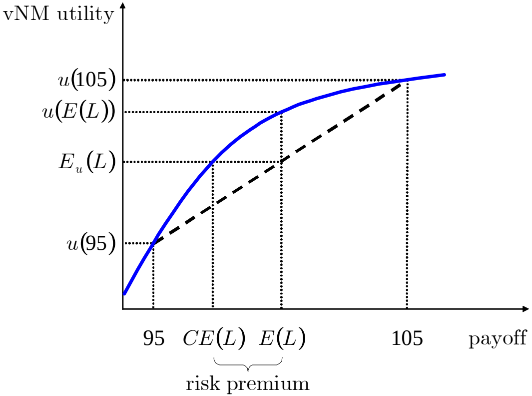
$$RP\left(L\right) := E\left(L\right) - CE\left(L\right)$$

– the risk premium.

Certainty equivalent and risk premium

Certainty equivalent

$$L = \left[95, 105; \tfrac{1}{2}, \tfrac{1}{2}\right]$$



Certainty equivalent and risk premium

Further exercises problem 1

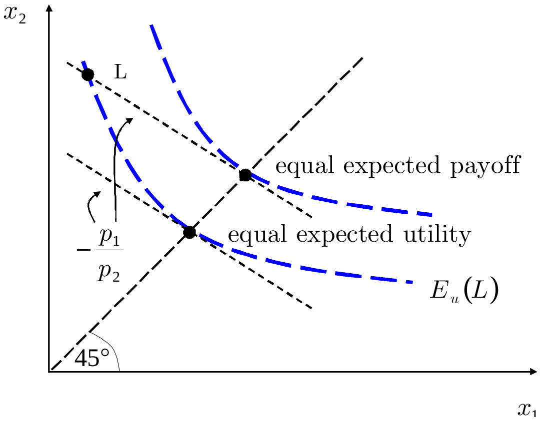
Problem

Reconsider the figure from the previous slide and draw a corresponding figure for risk neutral and risk-loving preferences.

Risk aversion and risk loving in an x1-x2-diagram

- $p := (p_1, ..., p_\ell)$ and $x := (x_1, ..., x_\ell)$;
- $E_u^p : \mathbb{R}_+^\ell \to \mathbb{R}$, $x \mapsto E_u^p(x) = E_u(x, p)$.

For $\ell = 2$:



Risk aversion and risk loving in an x1-x2-diagram

Slope of the indifference curve

$$
\begin{aligned}
MRS &= \frac{\frac{\partial E_u^p}{\partial x_1}}{\frac{\partial E_u^p}{\partial x_2}} = \frac{\frac{\partial [p_1 u(x_1) + p_2 u(x_2)]}{\partial x_1}}{\frac{\partial [p_1 u(x_1) + p_2 u(x_2)]}{\partial x_2}} = \frac{p_1 \frac{\partial u(x_1)}{\partial x_1}}{p_2 \frac{\partial u(x_2)}{\partial x_2}} \\
MRS &= \frac{p_1}{p_2} \text{ for } x_1 = x_2.
\end{aligned}
$$

Example

Risk neutrality:

$$
\begin{aligned}
u(x) &= ax + b, a > 0 \\
MRS(x_1) &= \frac{p_1 \frac{\partial u(x_1)}{\partial x_1}}{p_2 \frac{\partial u(x_2)}{\partial x_2}} = \frac{p_1 a}{p_2 a} = \frac{p_1}{p_2}
\end{aligned}
$$

Further exercises

Problem 1

Socrates has an endowment of 225 million Euro most of which is invested in a luxury cruise ship worth 200 million Euro. The ship sinks with a probability of $\frac{1}{5}$. Socrates vNM utility function is given by $u(x) = \sqrt{x}$. What is his willingness to pay for full insurance?

Problem 2

Identify the certainty equivalent and the risk premium in the x_1-x_2 diagram for risk-averse preferences.

Problem 3

Let $W = \{w_1, w_2\}$ be a set of 2 states of the world. The contingent good 1 that pays one Euro in case of state of the world w_1 and nothing in the other state is called an Arrow security. Determine this Arrow security in an x_1-x_2-diagram.

Further exercises: Problem 4

Sarah may become a paediatrician or a clerk in an insurance company. She expects to earn 40 000 Euro as a clerk every year. Her income as paediatrician depends on the number of children that will be born. In case of a baby boom, her yearly income will be 100 000 Euro, otherwise 20 000 Euro. She estimates the probability of a babyboom at $\frac{1}{2}$. Sarah's vNM utility function is given by $u(x) = 300 + \frac{4}{5}x$.

- Formulate Sarah's choices as lotteries!
- What is Sarah's choice?
- The Institute of Advanced Demography (IAD) has developed a secret, but reliable, method of predicting a baby boom. Sarah can buy the information for constant yearly rates. What is the maximum yearly willingness to pay?
- Sketch Sarah's decision problem in x_1-x_2 space where income without babyboom is noted at the x_1-axis and income with babyboom at the x_2-axis.