Advanced Microeconomics

Ordinal preference theory

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Part A. Basic decision and preference theory

1. Decisions in strategic (static) form
2. Decisions in extensive (dynamic) form
3. **Ordinal preference theory**
4. Decisions under risk

Ordinal preference theory

Overview

1. The vector space of goods and its topology
2. Preference relations
3. Axioms: convexity, monotonicity, and continuity
4. Utility functions
5. Quasi-concave utility functions and convex preferences
6. Marginal rate of substitution

The vector space of goods

Assumptions

- households are the only decision makers;
- finite number ℓ of goods defined by
  - place
  - time
  - contingencies
- example: an apple of a certain weight and class to be delivered
  - in Leipzig
  - on April 1st, 2015,
  - if it does not rain the day before

The vector space of goods

Exercise: Linear combination of vectors

Problem

Consider the vectors $x = (x_1, x_2) = (2, 4)$ *and* $y = (y_1, y_2) = (8, 12)$. *Find* $x + y$, $2x$ *and* $\frac{1}{4}x + \frac{3}{4}y$*!*

The vector space of goods

Notation

- For $\ell \in \mathbb{N}$:

$$\mathbb{R}^{\ell} := \{(x_1, ..., x_\ell) : x_g \in \mathbb{R}, g = 1, ..., \ell\}.$$

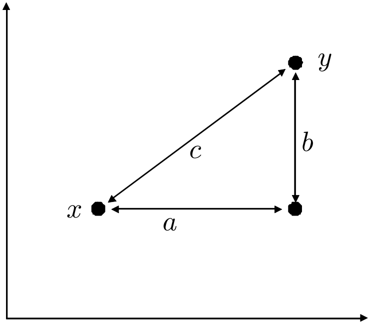
  - $0 \in \mathbb{R}^{\ell}$ – null vector $(0, 0, ..., 0)$;
  - vectors are called points (in $\mathbb{R}^{\ell}$).

- Positive amounts of goods only:

$$\mathbb{R}^{\ell}_{+} := \left\{x \in \mathbb{R}^{\ell} : x \geq 0\right\}$$

- Vector comparisons:
  - $x \geq y :\Leftrightarrow x_g \geq y_g$ for all g from $\{1, 2, ..., \ell\}$;
  - $x > y :\Leftrightarrow x \geq y$ and $x \neq y$;
  - $x \gg y :\Leftrightarrow x_g > y_g$ for all g from $\{1, 2, ..., \ell\}$.

Distance between x and y



euclidian
distance: c

city-block
distance: a+b

infinity
distance: max(a, b)

Distance

Definition

In $\mathbb{R}^\ell$:

- euclidian (or 2-) norm:
  $\|x - y\| := \|x - y\|_2 := \sqrt{\sum_{g=1}^{\ell} (x_g - y_g)^2}$
- infinity norm: $\|x - y\|_\infty := \max_{g=1,\dots,\ell} |x_g - y_g|$
- city-block norm: $\|x - y\|_1 := \sum_{g=1}^{\ell} |x_g - y_g|$

Distance

Exercise

Problem

What is the distance (in $\mathbb{R}^2$*) between* $(2, 5)$ *and* $(7, 1)$ *, measured by the* 2*-norm* $\|\cdot\|_2$ *and by the inifinity norm* $\|\cdot\|_\infty$*?*

Distance and balls

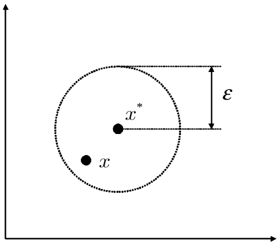
Ball

Definition

Let $x^* \in \mathbb{R}^\ell$ and $\varepsilon > 0$. $\Rightarrow$

$$K = \left\{ x \in \mathbb{R}^\ell : \|x - x^*\| < \varepsilon \right\}$$

– (open) ε-ball with center x^*.



- $\|x - x^*\| = \varepsilon$ holds for all x on the circular line;
- K – all the points within

Problem

Assuming the goods space $\mathbb{R}^2_+$, *sketch three* 1*-balls with centers* $(2, 2)$, $(0, 0)$ *and* $(2, 0)$, *respectively.*

Distance and balls

Boundedness

Definition

A set M is bounded if there exists an ε-ball K such that $M \subseteq K$.

Example

The set $[0, \infty) = \{x \in \mathbb{R} : x \geq 0\}$ is not bounded.

Sequences and convergence

Sequence

Definition

A sequence $\left(x^j\right)_{j \in \mathbb{N}}$ in $\mathbb{R}^\ell$ is a function $\mathbb{N} \to \mathbb{R}^\ell$.

Examples

- In $\mathbb{R}$: $1, 2, 3, 4, \dots$ or $x^j := j$;
- In $\mathbb{R}^2$:
  - $(1, 2), (2, 3), (3, 4), \dots$
  - $\left(1, \frac{1}{2}\right), \left(1, \frac{1}{3}\right), \left(1, \frac{1}{4}\right), \left(1, \frac{1}{5}\right), \dots$.

Definition

$\left(x^j\right)_{j\in\mathbb{N}}$ in $\mathbb{R}^\ell$ converges towards $x \in \mathbb{R}^\ell$ if for every $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that

$$\left\|x^j - x\right\| < \varepsilon \text{ for all } j > N$$

holds.

- Convergent – a sequence that converges towards some $x \in \mathbb{R}^\ell$.

Examples

- $1, 2, 3, 4, \ldots$ is not convergent towards any $x \in \mathbb{R}$;
- $1, 1, 1, 1, \ldots$ converges towards 1;
- $1, \frac{1}{2}, \frac{1}{3}, \ldots$ converges towards zero.

Lemma

Let $\left(x^j\right)_{j\in\mathbb{N}}$ *be a sequence in* $\mathbb{R}^\ell$.

- *If* $\left(x^j\right)_{j\in\mathbb{N}}$ *converges towards* x *and* y $\Rightarrow x = y$.
- $\left(x^j\right)_{j\in\mathbb{N}} = \left(x_1^j, \dots, x_\ell^j\right)_{j\in\mathbb{N}}$ *converges towards* $(x_1, \dots, x_\ell)$ *iff* x_g^j *converges towards* x_g *for every* $g = 1, \dots, \ell$.

Problem

Convergent?

$$(1,2), (1,3), (1,4), \dots$$

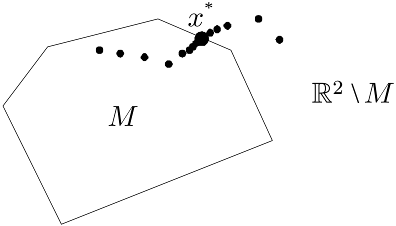
or

$$\left(1,\frac{1}{2}\right), \left(1,\frac{1}{3}\right), \left(1,\frac{1}{4}\right), \left(1,\frac{1}{5}\right), \dots .$$

Boundary point

Definition

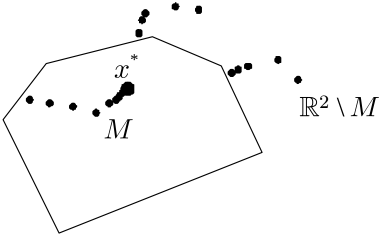
A point $x^* \in \mathbb{R}^\ell$ is a boundary point of $M \subseteq \mathbb{R}^\ell$ iff there is a sequence of points in M and another sequence of points in $\mathbb{R}^\ell \backslash M$ so that both converge towards x^*.



Interior point

Definition

A point in M that is not a boundary point is called an interior point of M.



Note: Instead of $\mathbb{R}^\ell$, we can consider alternative sets, for example $\mathbb{R}^\ell_+$.

Closed set

Definition

A set $M \subseteq \mathbb{R}^\ell$ is closed iff every converging sequence in M with convergence point $x \in \mathbb{R}^\ell$ fulfills $x \in M$.

Problem

Are the sets

- $\{0\} \cup (1, 2)$,
- $K = \left\{x \in \mathbb{R}^\ell : \|x - x^*\| < \varepsilon\right\}$,
- $K = \left\{x \in \mathbb{R}^\ell : \|x - x^*\| \leq \varepsilon\right\}$

closed?

Preference relations

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Definition

Relations R (write xRy) might be

- complete (xRy or yRx for all x, y)
- transitive (xRy and yRz implies xRz for all x, y, z)
- reflexive (xRx for all x);

Problem

For any two inhabitants from Leipzig, we ask whether one is the father of the other. Fill in "yes" or "no":

property	*is the father of*
reflexive	
transitive	
complete	

Definition

- (weak) preference relation $\precsim$ – a relation on $\mathbb{R}^{\ell}_{+}$ that is
  - complete ($x \precsim y$ or $y \precsim x$ for all x, y)
  - transitive ($x \precsim y$ and $y \precsim z$ implies $x \precsim z$ for all x, y, z) and
  - reflexive ($x \precsim x$ for all x);
- indifference relation:

$$x \sim y :\Leftrightarrow x \precsim y \text{ and } y \precsim x;$$

- strict preference relation:

$$x \prec y :\Leftrightarrow x \precsim y \text{ and not } y \precsim x.$$

Preference relations and indifference curves

Exercise

Problem

Fill in:

property	*indifference*	*strict preference*
reflexive		
transitive		
complete		

Preference relations and indifference curves

transitivity of strict preference

We want to show

$$x \prec y \wedge y \prec z \Rightarrow x \prec z$$

Proof: $x \prec y$ implies $x \precsim y$, $y \prec z$ implies $y \precsim z$.
Therefore, $x \prec y \wedge y \prec z$ implies $x \precsim z$.
Assume $z \precsim x$. Together with $x \prec y$, transitivity implies $z \precsim y$, contradicting $y \prec z$. Therefore, we do not have $z \precsim x$, but $x \prec z$.

Definition

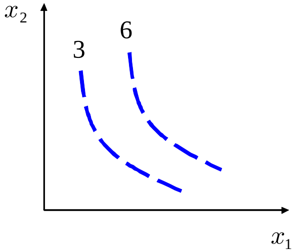
Let $\succsim$ be a preference relation on $\mathbb{R}_+^\ell$. $\Rightarrow$

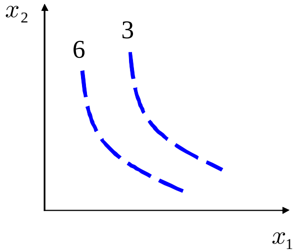
- $B_y := \left\{x \in \mathbb{R}_+^\ell : x \succsim y\right\}$ – better set B_y of y;
- $W_y := \left\{x \in \mathbb{R}_+^\ell : x \precsim y\right\}$ – worse set W_y of y;
- $I_y := B_y \cap W_y = \left\{x \in \mathbb{R}_+^\ell : x \sim y\right\}$ – y's indifference set;
- indifference curve – the geometric locus of an indifference set.

Preference relations and indifference curves

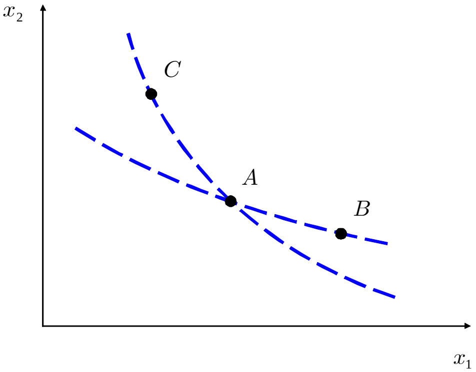
Indifference curve

numbers to indicate preferences:





Indifference curves must not intersect



Two different indifference curves,
Thus $C \nsim B$

But
$C \sim A \wedge A \sim B \Rightarrow C \sim B$

Contradiction!

Preference relations and indifference curves

Exercise

Problem

Sketch indifference curves for a goods space with just 2 goods and, alternatively,

- *good* 2 *is a bad,*
- *good* 1 *represents red matches and good* 2 *blue matches,*
- *good* 1 *stands for right shoes and good* 2 *for left shoes.*

Preference relations and indifference curves

Lexicographic preferences

In two-good case:

$$x \precsim_{lex} y :\Leftrightarrow x_1 < y_1 \text{ or } (x_1 = y_1 \text{ and } x_2 \leq y_2).$$

Problem

What do the indifference curves for lexicographic preferences look like?

Axioms: convexity, monotonicity, and continuity

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Convex preferences

Convex combination

Definition

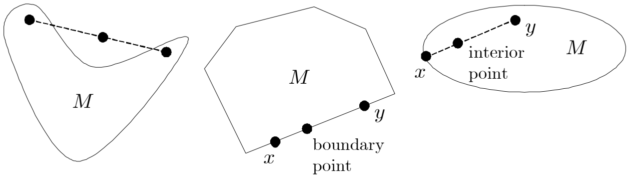
Let x and y be elements of $\mathbb{R}^{\ell}$. $\Rightarrow$

$$kx + (1 - k)\, y, k \in [0, 1]$$

– the convex combination of x and y.

Convex preferences

Convex and strictly convex sets



not convex

convex,
but not strictly convex

strictly convex

Definition

A set $M \subseteq \mathbb{R}^\ell$ is convex if for any two points x and y from M, their convex combination is also contained in M.

A set M is strictly convex if for any two points x and y from M, $x \neq y$,

$$kx + (1-k)\,y, k \in (0,1)$$

is an interior point of M for any $k \in (0,1)$.

Convex preferences

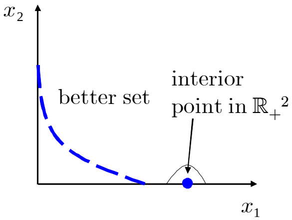
Convex and concave preference relation

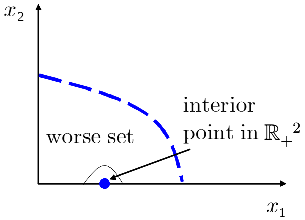
Definition

A preference relation $\succsim$ is

- convex if all its better sets B_y are convex,
- strictly convex if all its better sets B_y are strictly convex,
- concave if all its worse sets W_y are convex,
- strictly concave if all its worse sets W_y are strictly convex.

Preferences are defined on $\mathbb{R}^{\ell}_{+}$ (!):



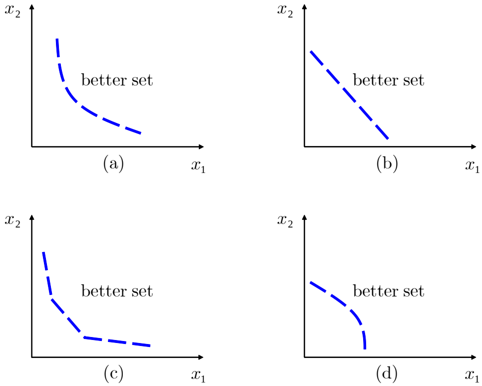


Convex preferences

Exercise

Problem

Are these preferences convex or strictly convex?



Monotonicity of preferences

Definition

A preference relation $\precsim$ obeys:

- weak monotonicity if $x > y$ implies $x \succsim y$;
- strict monotonicity if $x > y$ implies $x \succ y$;
- local non-satiation at y if in every ε-ball with center y a bundle x exists with $x \succ y$.

Problem

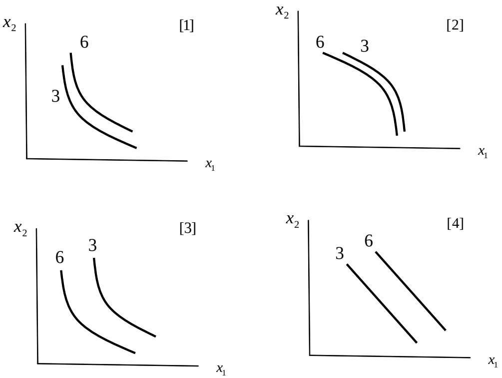
Sketch the better set of $y = (y_1, y_2)$ in case of weak monotonicity!

Exercise: Monotonicity and convexity

Which of the properties

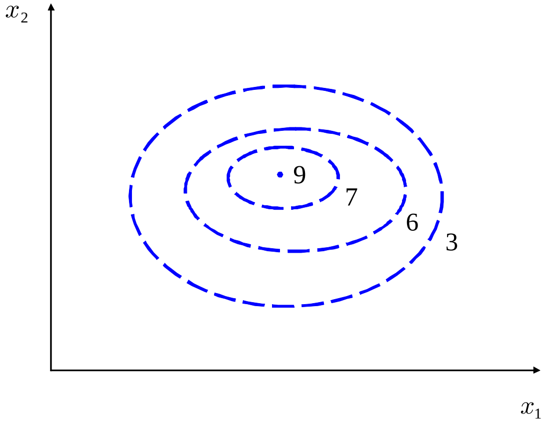
- (strict) monotonicity and/or
- (strict) convexity

do the preferences depicted by the indifference curves in the graphs below satisfy?



Monotonicity of preferences

Bliss point



Continuous preferences

Definition

A preference relation $\precsim$ is continuous if for all $y \in \mathbb{R}_+^\ell$ the sets

$$W_y = \left\{ x \in \mathbb{R}_+^\ell : x \precsim y \right\}$$

and

$$B_y = \left\{ x \in \mathbb{R}_+^\ell : y \precsim x \right\}$$

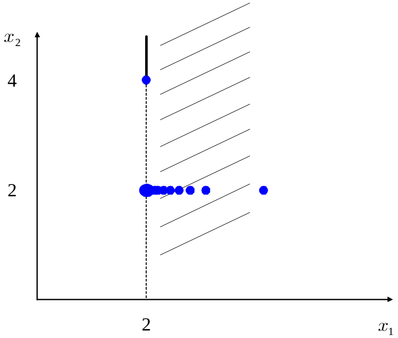
are closed.

Lexicographic preferences are not continuous

Consider the sequence

$$\left(x^j\right)_{j\in\mathbb{N}} = \left(2+\frac{1}{j},2\right) \to (2,2)$$

All its members belong to the better set of $(2,4)$.
But $(2,2)$ does not.



Utility functions

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Definition

For an agent $i \in N$ with preference relation $\succsim^i$,

$$U^i : \mathbb{R}_+^\ell \mapsto \mathbb{R}$$

– utility function if

$$U^i(x) \geq U^i(y) \Leftrightarrow x \succsim^i y, x, y \in \mathbb{R}_+^\ell$$

holds.

- U^i represents the preferences $\succsim^i$;
- ordinal utility theory.

Examples

- Cobb-Douglas utility functions (weakly monotonic):

$$U(x_1, x_2) = x_1^a x_2^{1-a} \text{ with } 0 < a < 1;$$

- perfect substitutes:

$$U(x_1, x_2) = ax_1 + bx_2 \text{ with } a > 0 \text{ and } b > 0;$$

- perfect complements:

$$U(x_1, x_2) = \min(ax_1, bx_2) \text{ with } a > 0 \text{ and } b > 0.$$

Utility functions

Dixit-Stiglitz preferences for love of variety

$$U\left(x_1, \ldots, x_\ell\right) = \left(\sum_{j=1}^{\ell} x_j^{\frac{\varepsilon-1}{\varepsilon}}\right)^{\frac{\varepsilon}{\varepsilon-1}} \quad \text{with } \varepsilon > 1$$

where $\bar{X} := \sum_{j=1}^{\ell} x_j$ implies

$$\begin{aligned} U\left(\frac{\bar{X}}{\ell}, \ldots, \frac{\bar{X}}{\ell}\right) &= \left(\sum_{j=1}^{\ell}\left(\frac{\bar{X}}{\ell}\right)^{\frac{\varepsilon-1}{\varepsilon}}\right)^{\frac{\varepsilon}{\varepsilon-1}} = \left(\sum_{j=1}^{\ell} \bar{X}^{\frac{\varepsilon-1}{\varepsilon}}\left(\frac{1}{\ell}\right)^{\frac{\varepsilon-1}{\varepsilon}}\right)^{\frac{\varepsilon}{\varepsilon-1}} \\ &= \left(\ell \bar{X}^{\frac{\varepsilon-1}{\varepsilon}}\left(\frac{1}{\ell}\right)^{\frac{\varepsilon-1}{\varepsilon}}\right)^{\frac{\varepsilon}{\varepsilon-1}} = \ell^{\frac{\varepsilon}{\varepsilon-1}} \bar{X} \frac{1}{\ell} = \ell^{\frac{\varepsilon}{\varepsilon-1}-1} \bar{X} = \ell^{\frac{1}{\varepsilon-1}} \bar{X} \end{aligned}$$

and hence, by $\frac{\partial U}{\partial \ell} > 0$, a love of variety.

Utility functions

Exercises

Problem

Draw the indifference curve for perfect substitutes with $a = 1$*,* $b = 4$ *and the utility level* 5*!*

Problem

Draw the indifference curve for perfect complements with $a = 1$*,* $b = 4$ *(a car with four wheels and one engine) and the utility level for* 5 *cars! Does* x_1 *denote the number of wheels or the number of engines?*

Equivalent utility functions

Definition (equivalent utility functions)

Two utility functions U and V are called equivalent if they represent the same preferences.

Lemma (equivalent utility functions)

Two utility functions U and V are equivalent iff there is a strictly increasing function $\tau : \mathbb{R} \to \mathbb{R}$ such that $V = \tau \circ U$.

Problem

Which of the following utility functions represent the same preferences?

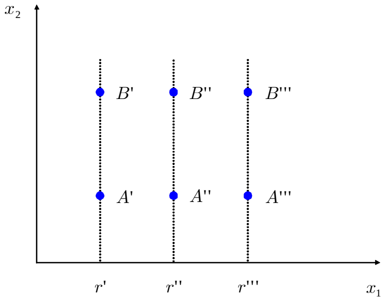
a) $U_1(x_1, x_2, x_3) = (x_1 + 1)(x_2 + 1)(x_3 + 1)$
b) $U_2(x_1, x_2, x_3) = \ln(x_1 + 1) + \ln(x_2 + 1) + \ln(x_3 + 1)$
c) $U_3(x_1, x_2, x_3) = -(x_1 + 1)(x_2 + 1)(x_3 + 1)$
d) $U_4(x_1, x_2, x_3) = -[(x_1 + 1)(x_2 + 1)(x_3 + 1)]^{-1}$
e) $U_5(x_1, x_2, x_3) = x_1 x_2 x_3$

Existence

Existence is not guaranteed

Assume a utility function U for lexicographic preferences:

- $U(A') < U(B') < U(A'') < U(B'') < U(A''') < U(B''')$;
- within $(U(A'), U(B'))$ at least one rational number (q') etc.
- $q' < q'' < q'''$;
- injective function $f : [r', r'''] \to Q$;
- not enough rational numbers;
- contradiction —> no utility function for lexicographic preferences

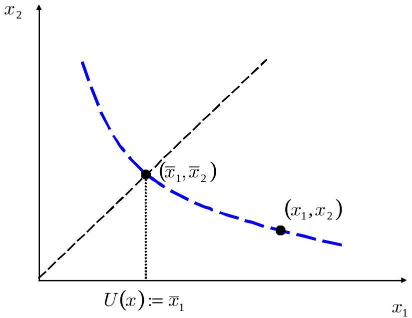


Existence

Existence of a utility function for continuous preferences

Theorem

If the preference relation $\precsim^i$ of an agent i is continuous, there is a continuous utility function U^i that represents $\precsim^i$.



Wait a second for the definition of a continuous function, please!

Existence

Problem

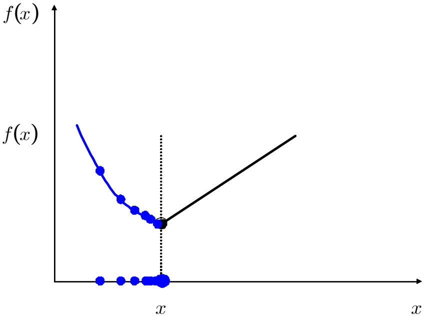
Assume a utility function U that represents the preference relation $\precsim$.
Can you express weak monotonicity, strict monotonicity and local non-satiation of $\precsim$ through U rather than $\precsim$?

Continuous functions

Definition

Definition

$f : \mathbb{R}^\ell \to \mathbb{R}$ is continuous at $x \in \mathbb{R}^\ell$ iff for every $\left(x^j\right)_{j\in\mathbb{N}}$ in $\mathbb{R}^\ell$ that converges towards x, $\left(f\left(x^j\right)\right)_{j\in\mathbb{N}}$ converges towards $f(x)$.



Fix $x \in \mathbb{R}^\ell$.

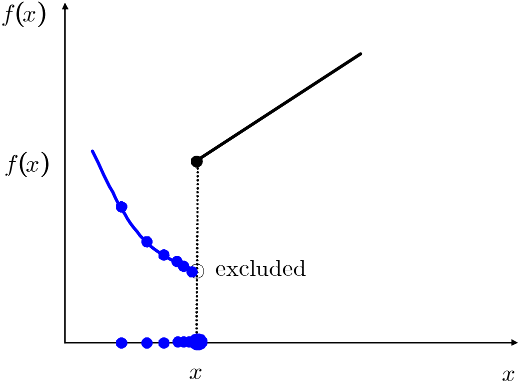
Consider sequences in the domain $\left(x^j\right)_{j\in\mathbb{N}}$ that converge towards x.

Convergence of sequences in the range $\left(f\left(x^j\right)\right)_{j\in\mathbb{N}}$ towards $f(x)$ —> f continuous at x

No convergence or convergence towards $y \neq f(x)$ —> no continuity

Continuous functions

Counterexample



Specific sequence in the domain $\left(x^j\right)_{j\in\mathbb{N}}$ that converges towards x but sequence in the range $\left(f\left(x^j\right)\right)_{j\in\mathbb{N}}$ does not converge towards $f\left(x\right)$

Quasi-concave utility functions and convex preferences

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Quasi-concavity

Definition

- $f : \mathbb{R}^\ell \to \mathbb{R}$ is quasi-concave if

$$f\left(kx + (1-k)\,y\right) \geq \min\left(f\left(x\right), f\left(y\right)\right)$$

  holds for all x, $y \in \mathbb{R}^\ell$ and all $k \in [0,1]$.

- f is strictly quasi-concave if

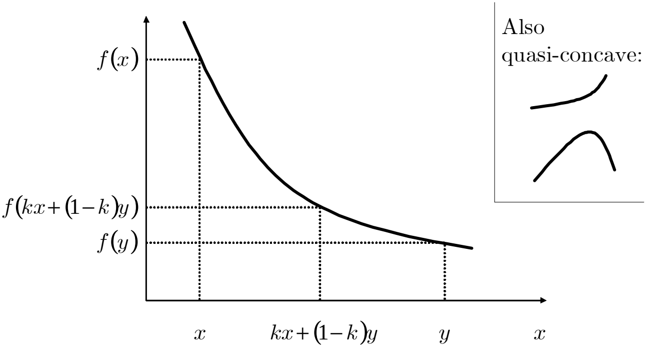
$$f\left(kx + (1-k)\,y\right) > \min\left(f\left(x\right), f\left(y\right)\right)$$

  holds for all x, $y \in \mathbb{R}^\ell$ with $x \neq y$ and all $k \in (0,1)$.

Note: quasi-concave functions need not be concave (to be introduced later).

Quasi-concavity

Examples

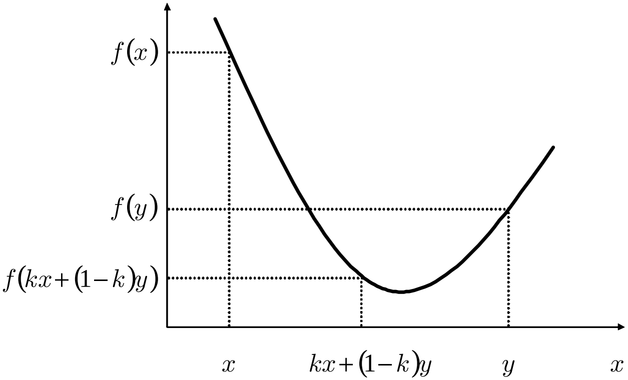


Example

Every monotonically increasing or decreasing function $f : \mathbb{R} \to \mathbb{R}$ is quasi-concave.

Quasi-concavity

A counter example



Better and worse sets, indifference sets

Definition

Let U be a utility function on $\mathbb{R}_+^\ell$.

- $B_{U(y)} := B_y = \left\{x \in \mathbb{R}_+^\ell : U(x) \geq U(y)\right\}$ – the better set B_y of y;
- $W_{U(y)} := W_y = \left\{x \in \mathbb{R}_+^\ell : U(x) \leq U(y)\right\}$ – the worse set W_y of y;
- $I_{U(y)} := I_y = B_y \cap W_y = \left\{x \in \mathbb{R}_+^\ell : U(x) = U(y)\right\}$ – y's indifference set (indifference curve) I_y.

Concave indifference curve

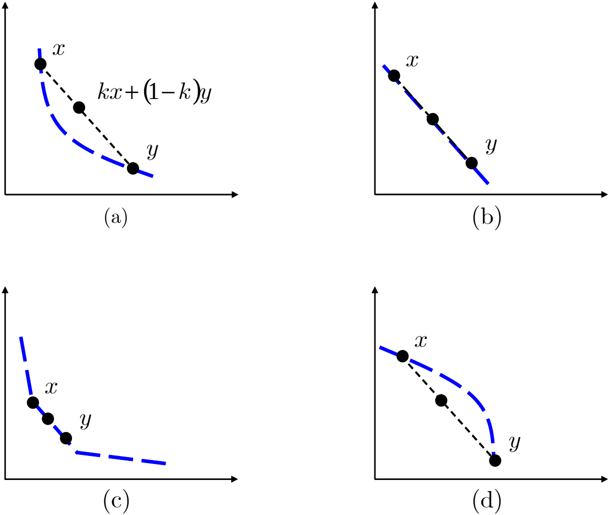
Definition

Let U be a utility function on $\mathbb{R}_+^\ell$.

- I_y is concave if $U(x) = U(y)$ implies
$$U(kx + (1-k)y) \geq U(x)$$
for all $x, y \in \mathbb{R}_+^\ell$ and all $k \in [0,1]$.
- I_y is strictly concave if $U(x) = U(y)$ implies
$$U(kx + (1-k)y) > U(x)$$
for all $x, y \in \mathbb{R}_+^\ell$ with $x \neq y$ and all $k \in (0,1)$.

Concave indifference curve

Examples and counter examples



Lemma

Lemma

Let U be a continuous utility function on $\mathbb{R}^{\ell}_{+}$. A preference relation $\succsim$ is convex iff:

- *all the indifference curves are concave, or*
- *U is quasi-concave.*

$$\begin{array}{ccccccc}
\begin{array}{l}U\text{'s better}\\ \text{sets strictly}\\ \text{convex}\end{array} & \Longleftarrow & \begin{array}{l}U\text{ strictly}\\ \text{quasi-concave}\end{array} & \Longrightarrow & \begin{array}{l}U\text{ quasi-}\\ \text{concave}\end{array} & \Longleftrightarrow & \begin{array}{l}U\text{'s better}\\ \text{sets convex}\end{array} \\
 & & \Updownarrow & & \Updownarrow & & \\
\begin{array}{l}U\text{'s better}\\ \text{sets strictly}\\ \text{convex and}\\ \text{local}\\ \text{nonsatiation}\end{array} & \Longrightarrow & \begin{array}{l}U\text{'s indifference}\\ \text{curves strictly}\\ \text{concave}\end{array} & \Longrightarrow & \begin{array}{l}U\text{'s indifference}\\ \text{curves concave}\end{array} & &
\end{array}$$

Marginal rate of substitution

Overview

1. The vector space of goods and its topology
2. Preference relations
3. Axioms: convexity, monotonicity, and continuity
4. Utility functions
5. Quasi-concave utility functions and convex preferences
6. **Marginal rate of substitution**

Definition

Let $f : M \to \mathbb{R}$ be a real-valued function with open domain $M \subseteq \mathbb{R}^\ell$.
f is differentiable if all the partial derivatives

$$f_i(x) := \frac{\partial f}{\partial x_i} \ (i = 1, ..., \ell)$$

exist and are continuous.

$$f'(x) := \begin{pmatrix} f_1(x) \\ f_2(x) \\ ... \\ f_\ell(x) \end{pmatrix}$$

– f's derivative at x.

Marginal rate of substitution

Mathematics: Adding rule

Theorem

Let $f : \mathbb{R}^{\ell} \rightarrow \mathbb{R}$ *be a differentiable function and let* $g_1, ..., g_{\ell}$ *be differentiable functions* $\mathbb{R} \rightarrow \mathbb{R}$*. Let* $F : \mathbb{R} \rightarrow \mathbb{R}$ *be defined by*

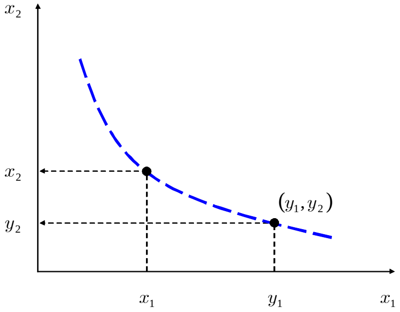
$$F(x) = f(g_1(x), ..., g_{\ell}(x)).$$

$\Rightarrow$

$$\frac{dF}{dx} = \sum_{i=1}^{\ell} \frac{\partial f}{\partial g_i} \frac{dg_i}{dx}.$$

Marginal rate of substitution

Economics



- $I_y = \left\{(x_1, x_2) \in \mathbb{R}^2_+ : (x_1, x_2) \sim (y_1, y_2)\right\}.$
- $I_y : x_1 \mapsto x_2.$

Marginal rate of substitution

Definition

If the function I_y is differentiable and if preferences are monotonic,

$$\left| \frac{dI_y\left(x_1\right)}{dx_1} \right|$$

– the MRS between good 1 and good 2 (or of good 2 for good 1).

Problem

What happens if good 2 is a bad?

Marginal rate of substitution

Perfect substitutes

Problem

Calculate the MRS for perfect substitutes ($U(x_1, x_2) = ax_1 + bx_2$ with $a > 0$ and $b > 0$.)!

- Solve $ax_1 + bx_2 = k$ for x_2!
- Form the derivative of x_2 with respect to x_1!
- Take the absolute value!

Marginal rate of substitution

Lemma 1

Lemma

Let $\succsim$ be a preference relation on $\mathbb{R}^{\ell}_{+}$ and let U be the corresponding utility function.

If U is differentiable, the MRS is defined by:

$$MRS\left(x_1\right) = \left|\frac{dI_y\left(x_1\right)}{dx_1}\right| = \frac{\frac{\partial U}{\partial x_1}}{\frac{\partial U}{\partial x_2}}.$$

$\frac{\partial U}{\partial x_1}, \frac{\partial U}{\partial x_2}$ – marginal utility.

Marginal rate of substitution

Lemma: proof

Proof.

- $U(x_1, I_y(x_1))$ – constant along indifference curve;
- differentiating $U(x_1, I_y(x_1))$ with respect to x_1 (adding rule):

$$0 = \frac{\partial U}{\partial x_1} + \frac{\partial U}{\partial x_2} \frac{dI_y(x_1)}{dx_1}.$$

- $\left| \frac{dI_y(x_1)}{dx_1} \right|$ can be found even if I_y were not given explicitly (implicit-function theorem).

$\square$

Problem

Again: What is the MRS in case of perfect substitutes?

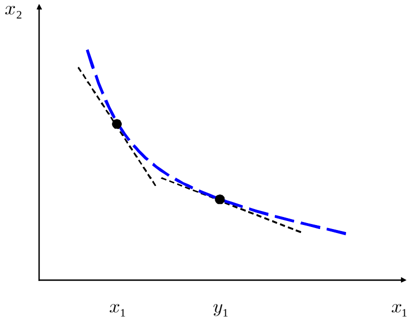
Marginal rate of substitution

Lemma 2

Lemma

Let U be a differentiable utility function and I_y an indifference curve of U.

I_y is concave iff the MRS is a decreasing function in x_1.



- $x_1 < y_1$;
- $MRS\,(x_1) > MRS\,(y_1)$.

Marginal rate of substitution

Cobb-Douglas utility function

$$U\left(x_1, x_2\right) = x_1^a x_2^{1-a}, 0 < a < 1$$

$$MRS = \frac{\frac{\partial U}{\partial x_1}}{\frac{\partial U}{\partial x_2}} = \frac{a x_1^{a-1} x_2^{1-a}}{\left(1-a\right) x_1^a x_2^{-a}} = \frac{a}{1-a}\frac{x_2}{x_1}.$$

- Cobb-Douglas preferences are monotonic so that an increase of x_1 is associated with a decrease of x_2 along an indifference curve.
- Therefore, Cobb-Douglas preferences are convex (Cobb-Douglas utility functions are quasi-concave).

Further exercises

Problem 1

Define strict anti-monotonicity. Sketch indifference curves for each of the four cases:

	strict monotonicity	strict anti-monotonicity
strict convexity		
strict concavity		

Problem 2

(Strictly) monotonic, (strictly) convex or continuous?

(a) $U(x_1, x_2) = x_1 \cdot x_2$,

(b) $U(x_1, x_2) = \min\{a \cdot x_1, b \cdot x_2\}$ where $a, b > 0$ holds,

(c) $U(x_1, x_2) = a \cdot x_1 + b \cdot x_2$ where $a, b > 0$ holds,

(d) lexicographic preferences

Further exercises

Problem 3 (difficult)
Let U be a continuous utility function representing the preference relation $\precsim$ on $\mathbb{R}_+^\ell$. Show that $\precsim$ is continuous as well. Also, give an example for a continuous preference relation that is represented by a discontinuous utility function. Hint: Define a function U' that differs from U for $x = 0$, only