Advanced Microeconomics

Decisions in extensive form

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Part A. Basic decision and preference theory

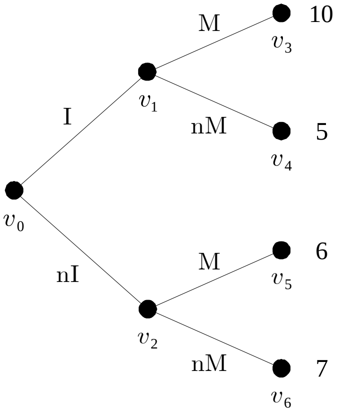
1. Decisions in strategic (static) form
2. **Decisions in extensive (dynamic) form**
3. Ordinal preference theory
4. Decisions under risk

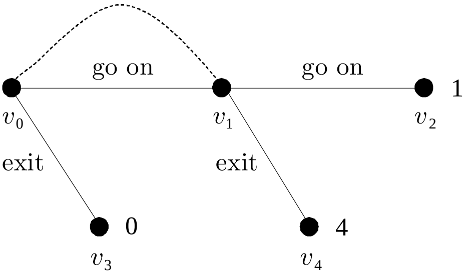
1. **Introduction**
2. Strategies and subtrees: perfect information
3. Strategies and subtrees: imperfect information
4. Moves by nature, imperfect information and perfect recall

Example

Two-stage decision situation of an umbrella-producing firm:

- stage 1: action I (investing) or action nI (no investment);
- stage 2: action M (marketing activities) or nM (no marketing activities).





Two different sorts of nodes:

- I have to make a decision.
- I get something.

Strategies and subtrees: perfect information

Overview

Decision situation in extensive form: perfect information

Partition, component, singleton

Definition

M nonempty set. $\mathcal{P}_M = \{M_1, ..., M_k\}$ is a partition of M if

$$\begin{aligned} \bigcup_{j=1}^{k} M_j &= M \text{ and} \\ M_j \cap M_\ell &= \emptyset \text{ for all } j, \ell \in \{1, ..., k\}, j \neq \ell \end{aligned}$$

- component (normally nonempty) – element of a partition;
- $\mathcal{P}_M(m)$ – component containing m;
- singleton – component with one element only;

Problem

Write down two partitions of $M := \{1, 2, 3\}$. Find $\mathcal{P}_M(1)$ in each case.

Perfect-information decision situation

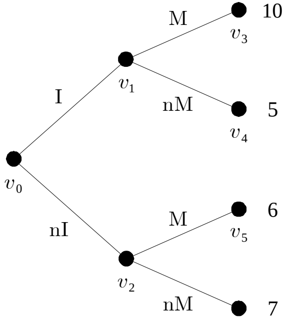
Decision situation $\Delta =$

- Tree with nodes, often denoted by v_0, v_1, ...
- Initial node v_0 and exactly one trail
  initial node —> specific end node
- Decision nodes D:
  Nodes at which actions can be taken
  $=$ non-terminal nodes
- Actions A_d at $d \in D$ and union A
- Terminal nodes $=$ end nodes E :
  with payoff information
- $D \cup E = V$: set of all nodes

Length of trail $\langle v_0, v_3 \rangle$: 2, length of trail $\langle v_1, v_3 \rangle$: 1
Length of tree: maximal length of any trail

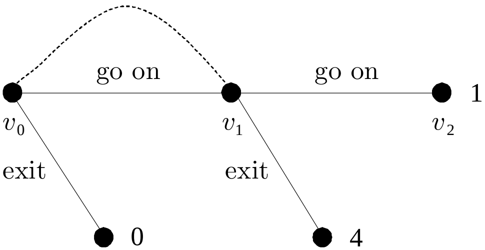
Problem

What is the length of this tree?



Problem

What is the length of this tree?



Imperfect-information decision situation

Decision situation Δ with nodes from D and E

Definition

- I (information partition) $\rightarrow$ a partition of the decision nodes D;
- Elements of I are called information sets (which are components)
- For some $d \in D$: $I(d) = \{d\}$ (the decision maker knows where he is)
- For others: we have $I(d) = I(d') = \{d, d', ...\}$
- $A_d = A_{d'}$ for all $d, d' \in I(d)$

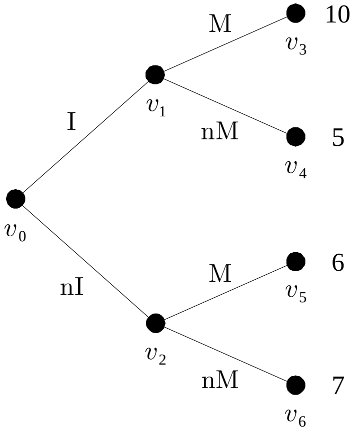
Problem

For the absent-minded driver, specify $I(v_0)$ and A_{v_0}? How about A_{v_1}?

Strategies
Example

- We define a strategy by
  - $s(v_0) = \text{nI}$
  - $s(v_1) = \text{M}$
  - $s(v_2) = \text{nM}$

  that can also be written as $\lfloor \text{nI}, \text{M}, \text{nM} \rfloor$.
- How many strategies do we have?
- Which strategies are best?
- What is the difference between
  - $\lfloor \text{I}, \text{M}, \text{M} \rfloor$ and
  - $\lfloor \text{I}, \text{M}, \text{nM} \rfloor$?



Strategies

definition

Definition

A strategy is a function $s : D \to A$ where A is the set of actions and $s(d) \in A_d$ for all $d \in D$.

Problem

What does $|S| = \prod_{d \in D} |A_d|$ *mean? Is it correct?*

Definition

A strategy $s \in S$ can provoke a node v or a trail $\langle v_0, v_1, ..., v_k = v \rangle$ (defined in the obvious manner). The terminal node provoked by strategy s is denoted by v_s.
Define:

$$\begin{aligned} u(s) &: = u(v_s), s \in S \text{ and} \\ s^R(\Delta) &: = \arg\max_{s \in S} u(s). \end{aligned}$$

Problem

Indicate all the nodes provoked by the strategy $\lfloor I, M, M \rfloor$ in the investment-marketing example. Which strategies are best?

Strategies

definition: why so complete?

Two reasons:

- simple definition
- we want to distinguish between
  - $\lfloor \mathrm{I}, \mathrm{M}, \mathrm{M} \rfloor$ and
  - $\lfloor \mathrm{I}, \mathrm{M}, \mathrm{nM} \rfloor$

  above.

Definition

Let $f : X \to Y$ be a function.
For $X' \subseteq X$,

$$f|_{X'} : X' \to Y$$

is a restriction of f to X' if $f|_{X'}(x) = f(x)$ holds for all $x \in X'$.

Consider a decision node $w \in D$.
w and the nodes following w make up the set W.

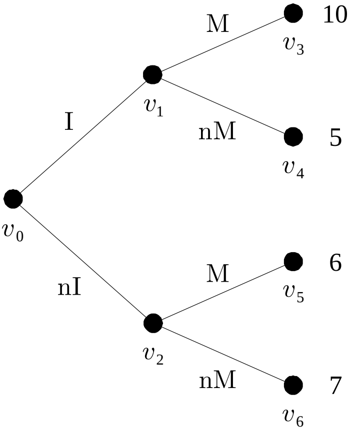
- Decision situation Δ^w generated from decision situation Δ
  $=$ subtree
- strategy s^w (in Δ^w) generated from strategy s (in Δ)
  $=$ the restriction of s to $W \cap D$.
  $s^w = s|_{W \cap D}$

Three subtrees that begin at

- v_0 $(\Delta^{v_0} = \Delta)$
- v_1
- v_2

Strategy $s = \lfloor \text{nI, M, nM} \rfloor$ generates strategies

- $s^{v_1} = \lfloor \text{M} \rfloor$ in Δ^{v_1}
- $s^{v_2} = \lfloor \text{nM} \rfloor$ in Δ^{v_2}



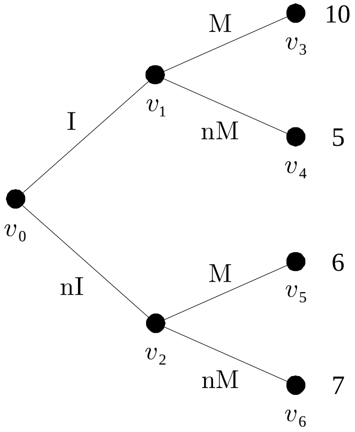
Definition

A strategy s is subtree-perfect if, for every $w \in D$, s^w is a best strategy in the decisional subtree Δ^w.

Problem

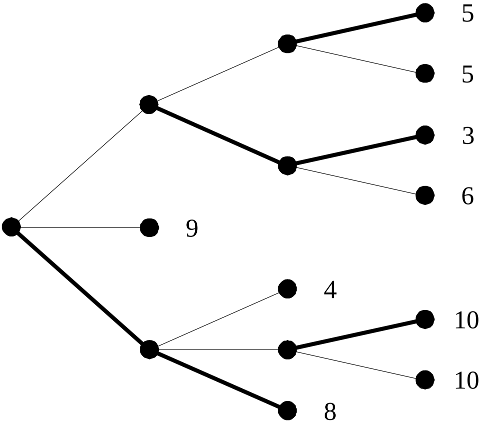
Are these strategies subtree-perfect:

- $\lfloor nI, M, nM \rfloor$,
- $\lfloor I, M, M \rfloor$,
- $\lfloor I, M, nM \rfloor$



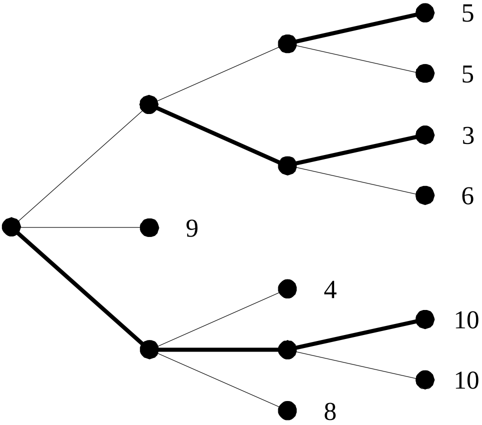
Problem

Optimal strategy?
Subtree-perfect strategy?



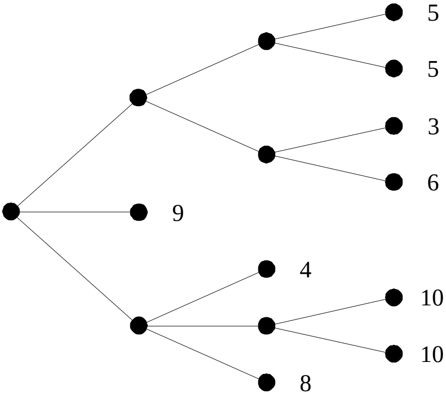
Problem

Optimal strategy?
Subtree-perfect strategy?



Backward induction for perfect information

Exercise

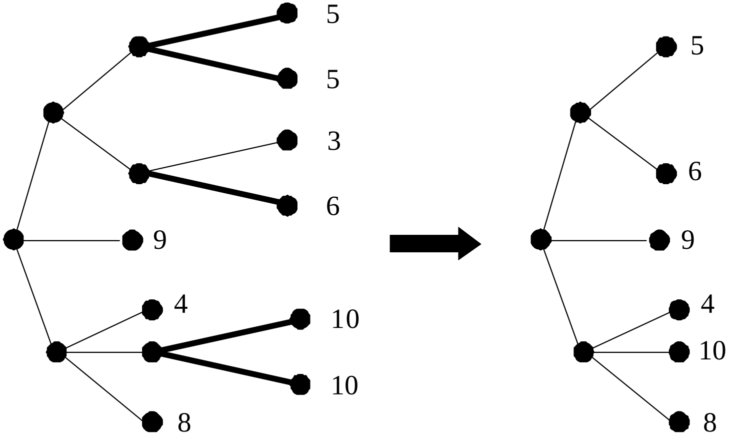


Backward induction means

- starting with the smallest subtrees,
- noting the best actions,
- and working towards the initial node

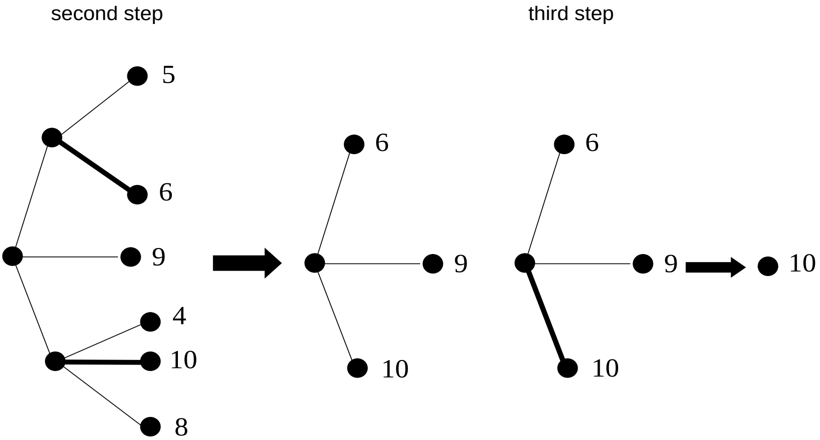
Backward induction for perfect information

Solution: first step



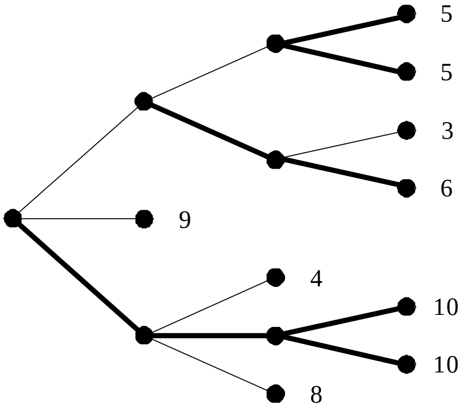
Backward induction for perfect information

Solution: second and third step



Backward induction for perfect information

without drawing several trees



- Backward-induction trails (how many?) versus
- backward-induction strategies (how many?)

Subtree perfection and backward induction for perfect information

Theorem

If Δ is of finite length (trails do not go on forever), the set of subtree-perfect strategies and the set of backward-induction strategies coincide.

Thus, you can find all subtree-perfect strategies by applying backward induction.

Definition (Transitivity axiom)

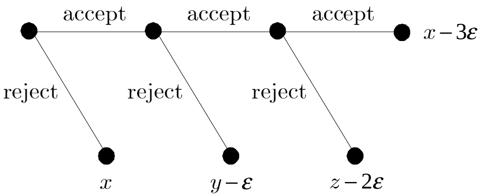
If a person prefers x to y and y to z then she should also prefer x to z.

The money-pump argument supports the transitivity axiom. Assume prefrences are not transitive:

- $x \prec y \prec z \prec x$;
- agent starts with x;
- agent exchanges:
  - x against y and offers ε $(x \prec y - \varepsilon)$ —> $y - \varepsilon$.
  - y against z and offers ε $(y \prec z - \varepsilon)$ —> $z - 2\varepsilon$.
  - z against x and offers ε $(z \prec x - \varepsilon)$ —> $x - 3\varepsilon$.
- agent ends up with $x - 3\varepsilon$

The money pump

The decision tree



8 strategies:

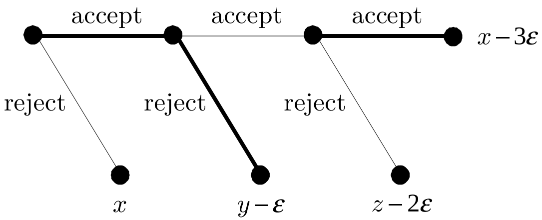
$$\lfloor \text{accept, accept, accept} \rfloor,$$
$$\lfloor \text{accept, reject, accept} \rfloor \text{ and}$$
$$\lfloor \text{reject, accept, reject} \rfloor$$

Problem

Write down all strategies that lead to payoff $y - \varepsilon$.

Assume $x \prec y \prec z \prec x$ and also

- $x \prec y - \varepsilon \prec z - 2\varepsilon \prec x - 3\varepsilon$ and
- $x - 3\varepsilon \prec y - \varepsilon$



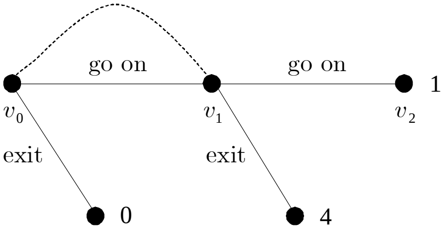
Backward induction does not support the money-pump argument!

Strategies and subtrees: imperfect information
Overview

1. Introduction
2. Decision trees and actions
3. Strategies and subtrees: perfect information
4. **Strategies and subtrees: imperfect information**
5. Moves by nature, imperfect information and perfect recall

Strategies and subtrees: imperfect information

Example



Definition

A strategy is a function $s : D_1 \to A$ with

- $s(d) \in A_d$ for all $d \in D$ and
- $s(d) = s(d')$ for all $d, d' \in I(d)$.

Problem

Strategies? Best strategies?

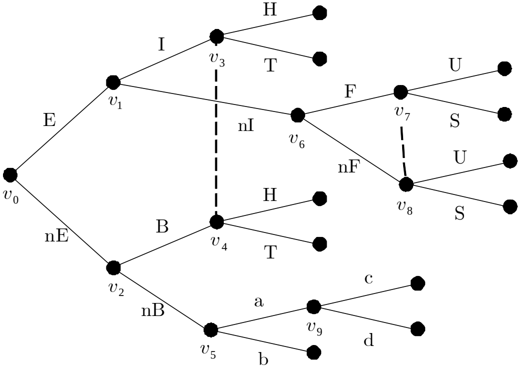
Consider a decision node $w \in D$.
w and the nodes following w make up the set W.

- Decision situation Δ^w generated from decision situation Δ
  $=$ subtree
  if W does not cut into an information set
  i.e., if there is no information set that belongs to W and to $V \backslash W$ at the same time
- strategy s^w (in subtree Δ^w) generated from strategy s (in Δ)
  $=$ the restriction of s to $W \cap D$
  $s^w = s|_{W \cap D}$

Go back to absent-minded driver and check for subtrees!

Strategies and subtrees: imperfect information

Problem



Subtrees?
How many strategies?
One example!

Strategies and subtrees: imperfect information

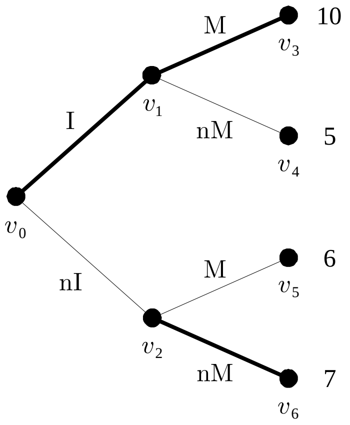
Nodes provoked by s

Problem

Consider the mixed strategy σ given by

$$\sigma(s) = \begin{cases} \frac{1}{3}, & s = \lfloor I,\ M,\ nM \rfloor \\ \frac{1}{6}, & s = \lfloor nI,\ M,\ nM \rfloor \\ \frac{1}{12}, & s \text{ otherwise} \end{cases}$$

Is σ well-defined?
What is the probability for node v_3?



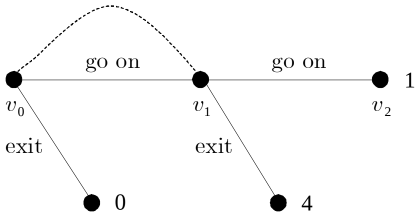
Behavioral strategies

Definition

Decision situation of imperfect information Δ

- $\beta = (\beta_d)_{d \in D}$ - tuple of probability distributions, where β_d is a probability distribution on A_d that obeys $\beta_d = \beta_{d'}$ for all $d, d' \in I(d)$
- β is called a behavioral strategy.

Best behavioral strategy for the absent-minded driver?



- $\beta_{exit} := \beta_{v_0}(exit)$ – the probability for exit;
- expected payoff:

$$\underbrace{\beta_{exit}}_{\substack{\text{exit probability} \\ \text{at } v_0}} \cdot 0 + \underbrace{(1-\beta_{exit})\,\beta_{exit}}_{\substack{\text{exit probability} \\ \text{at } v_1}} \cdot 4 + \underbrace{(1-\beta_{exit})\,(1-\beta_{exit})}_{\substack{\text{probability} \\ \text{of going on}}} \cdot 1$$

$$= \quad -3\beta_{exit}^2 + 2\beta_{exit} + 1$$

- optimal behavioral strategy:

$$\beta_{exit}^* = \arg\max_{\beta_{exit}} \left(-3\beta_{exit}^2 + 2\beta_{exit} + 1\right) = \frac{1}{3}.$$

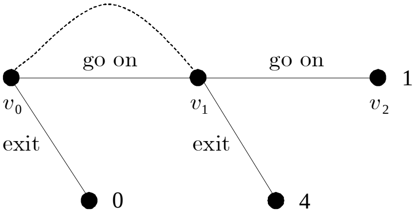
Experience

Definition

At $v \in D$, the experience $X(v)$ is the sequence (tuple) of information sets and actions at these information sets as they occur along the trail from v_0 to v. An information set is the last entry of an experience.

Absent-minded driver example:

- $X(v_0) = (I(v_0))$ and
- $X(v_1) = (I(v_0), \text{ go on}, I(v_1))$.



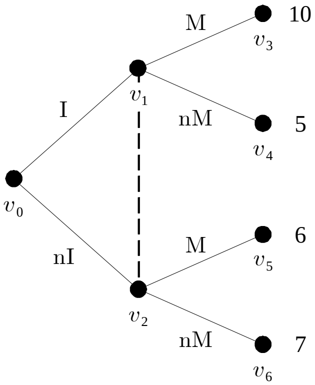
Experience and perfect recall

Definition

Δ is characterized by perfect recall if for all $v, v' \in D$ with $I(v) = I(v')$ we have $X(v) = X(v')$.

Problem

Does perfect information imply perfect recall?



Problem

Show that this decision situation exhibits imperfect recall!

Problem

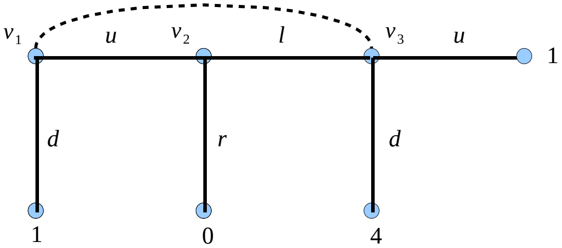
Strategies $\lfloor I, M \rfloor$ and $\lfloor I, nM \rfloor$ provoke which nodes?

- Shouldn't the strategy $\lfloor$I, M$\rfloor$ tell the decision maker that he is at v_1?
- Interpretation: magic ink that disappears after use

Summary

- perfect information $\longrightarrow$ every information set contains only one element;
- (properly) imperfect information $\longrightarrow$ there is at least one information set with more than one element;
- perfect recall $\longrightarrow$ in every information set, all decision nodes are associated with the same "experience";
- imperfect recall $\longrightarrow$ two decision nodes exist that belong to the same information set but result from different "experiences".

Exercise



(a) Pure strategies, information sets, proper subtrees?

(b) Optimal mixed strategies?

(c) Perfect recall?

(d) Optimal behavioral strategies?

Theorem

Decision situation with perfect recall.
A given probability distribution on the set of terminal nodes is achievable by a mixed strategy iff it is achievable by a behavioral strategy (payoff equivalence).

Kuhn's theorem continues to hold when moves by nature are included (see following section).

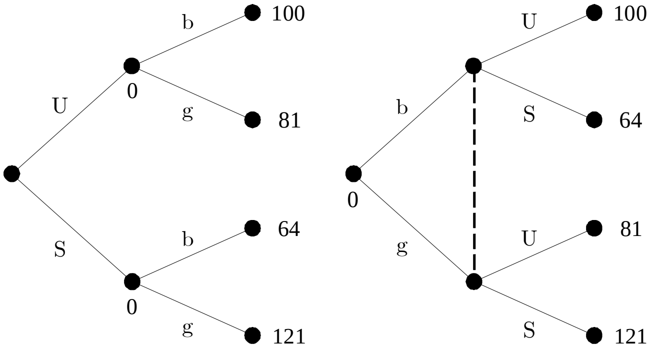
Moves by nature

Overview

1. Introduction
2. Decision trees and actions
3. Strategies and subtrees: perfect information
4. Strategies and subtrees: imperfect information
5. **Moves by nature, imperfect information and perfect recall**

Moves by nature

Uncertainty about the weather



- nature is denoted by "0";
- uncertainty about the weather in both cases but
  - perfect information in the left-hand tree;
  - **imperfect information in the right-hand tree with nature moving first**

Moves by nature

- Nature:
  D_0 = nature's decision nodes
  $\beta_0 \rightarrow$ tuple of probability distributions $(\beta_d)_{d \in D_0}$ on A_0 (actions choosable by nature)
- Decider:
  D_1 = decision maker's decision nodes
  The information partition I partitions D_1!

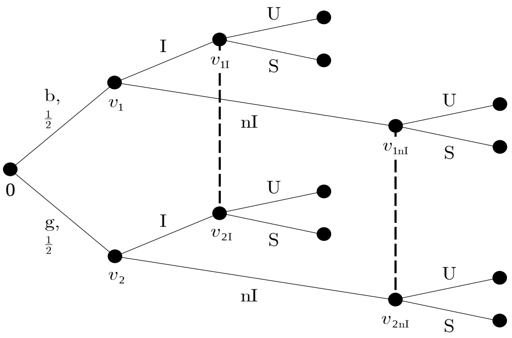
A strategy is a function $s : D_1 \rightarrow A$ with feasible actions where $s(d) = s(d')$ for all $d, d' \in I(d)$.

Subtrees may start at nodes from $D = D_0 \cup D_1$.
In particular, the whole tree is always a subtree of itself.

Experience defined at $v \in D_1$! In case of $v' \in D_0$, $I(v')$ is not defined!

Problem

Indicate the probability distributions on the set of terminal nodes provoked by the strategies $\lfloor I, nI, S, U \rfloor$ and by $\lfloor nI, nI, S, S \rfloor$ by writing the probabilities near these nodes!

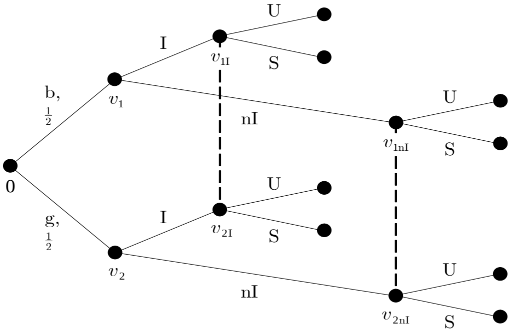


Moves by nature

Exercise 2

Problem

Does this decision tree reflect perfect or imperfect recall? How many subtrees can you identify?

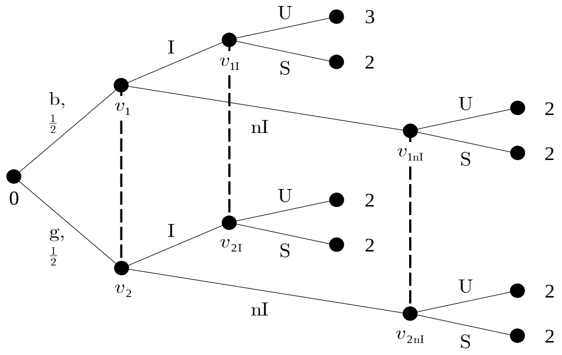


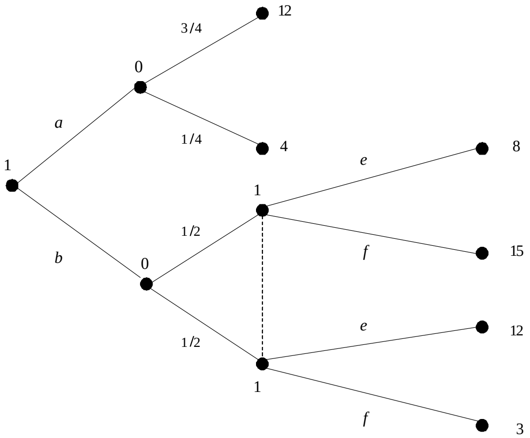
Moves by nature

Exercise 3

Problem

Does this decision tree reflect perfect or imperfect recall? How many subtrees can you identify?

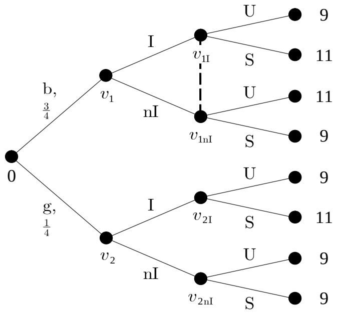




(a) Pure strategies?

(b) Perfect recall?

(c) Optimal strategies?

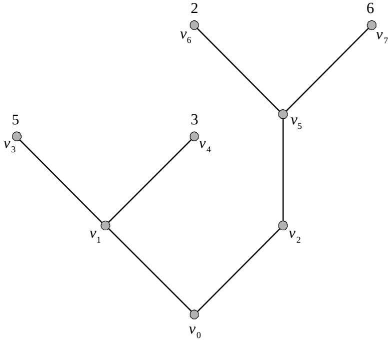


Backward induction means

- starting with the smallest subtrees,
- noting the best substrategies,
- and working towards the initial node

(a) subtrees?
(b) backward-induction strategies?
(c) subtree-perfect strategies?
(d) perfect recall?

Further exercises: Problem 1

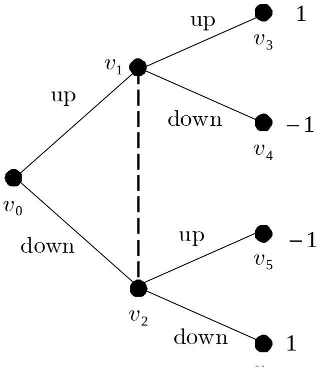


(a) How many subtrees?

(b) How many strategies? Which are the best?

(c) Backward induction!

Further exercises: Problem 2



(a) True or false? In this decision situation, any behavioral strategy can be characterized by specifying two probabilities.

(b) Perfect recall?

(c) Best mixed strategy and the best behavioral strategy?