# Advanced Microeconomics

## Comparative statics and duality theory

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# Part B. Household theory and theory of the firm

1. The household optimum
2. **Comparative statics and duality theory**
3. Production theory
4. Cost minimization and profit maximization

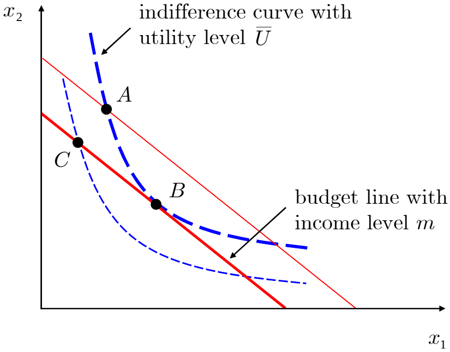
# Comparative statics and duality theory

## Overview

1. The duality approach
2. Shephard's lemma
3. The Hicksian law of demand
4. Slutsky equations
5. Compensating and equivalent variations

# Maximization and minimization problem

- Maximization problem: Find the bundle that maximizes the utility for a given budget line.
- Minimization problem: Find the bundle that minimizes the expenditure needed to achieve a given utility level.



# The expenditure function and the Hicksian demand function I

Expenditure function:

$$\begin{aligned} e &: \mathbb{R}^\ell \times \mathbb{R} \to \mathbb{R}, \\ &(p, \bar{U}) \mapsto e(p, \bar{U}) := \min_{\substack{x \text{ with} \\ U(x) \geq \bar{U}}} px \end{aligned}$$

The solution to the minimization problem is called the Hicksian demand function:

$$\begin{aligned} \chi &: \mathbb{R}^\ell \times \mathbb{R} \to \mathbb{R}^\ell_+, \\ &(p, \bar{U}) \mapsto \chi(p, \bar{U}) := \arg \min_{\substack{x \text{ with} \\ U(x) \geq \bar{U}}} px \end{aligned}$$

# The expenditure function and the Hicksian demand function II

## Problem

*Express*

- *$e$ in terms of $\chi$ and*
- *$V$ in terms of the household optima!*

## Lemma

*For any $\alpha > 0$:*

$$\chi\left(\alpha p, \bar{U}\right) = \chi\left(p, \bar{U}\right) \text{ and } e\left(\alpha p, \bar{U}\right) = \alpha e\left(p, \bar{U}\right).$$

Obvious?!

# The expenditure function and the Hicksian demand function III

## Problem

*Determine the expenditure function and the Hicksian demand function for the Cobb-Douglas utility function* $U(x_1, x_2) = x_1^a x_2^{1-a}$ *with* $0 < a < 1$*!*
*Hint: We know that the indirect utility function* $V$ *is given by:*

$$\begin{aligned} V(p, m) &= U(x(p, m)) \\ &= \left(a\frac{m}{p_1}\right)^a \left((1-a)\frac{m}{p_2}\right)^{1-a} \\ &= \left(\frac{a}{p_1}\right)^a \left(\frac{1-a}{p_2}\right)^{1-a} m. \end{aligned}$$

# Hicksian demand and the expenditure function

| | utility maximization | expenditure minimization |
|---|---|---|
| objective function | utility | expenditure |
| parameters | prices $p$, income $m$ | prices $p$, utility $\bar{U}$ |
| notation for best bundle(s) | $x(p,m)$ | $\chi(p,\bar{U})$ |
| name of demand function | Marshallian | Hicksian |
| value of objective function | $V(p,m)$ $= U(x(p,m))$ | $e(p,\bar{U})$ $= p \cdot \chi(p,\bar{U})$ |

# Applying the Lagrange method (recipe)

$$L(x,\mu) = \sum_{g=1}^{\ell} p_g x_g + \mu \left[\bar{U} - U(x)\right]$$

with:

- $U$ – strictly quasi-concave and strictly monotonic utility function;
- prices $p >> 0$;
- $\mu > 0$ – Lagrange multiplier - translates a utility surplus (in case of $U(x) > \bar{U}$) into expenditure reduction
- Increasing consumption has
  - a positive direct effect on expenditure, but
  - a negative indirect effect via a utility surplus (scope for expenditure reduction) and $\mu$

# Applying the Lagrange method (recipe)

Differentiate $L$ with respect to $x_g$ :

$$\frac{\partial L\left(x_1, x_2, ..., \mu\right)}{\partial x_g} = p_g - \mu \frac{\partial U\left(x_1, x_2, ..., x_\ell\right)}{\partial x_g} \overset{!}{=} 0 \text{ or } \frac{\partial U\left(x_1, x_2, ..., x_\ell\right)}{\partial x_g} \overset{!}{=} \frac{p_g}{\mu}$$

and hence, for two goods $g$ and $k$

$$\frac{\frac{\partial U(x_1,x_2,...,x_\ell)}{\partial x_g}}{\frac{\partial U(x_1,x_2,...,x_\ell)}{\partial x_k}} \overset{!}{=} \frac{p_g}{p_k} \text{ or } MRS \overset{!}{=} MOC$$

Note also: $\frac{\partial L(x,\mu)}{\partial \mu} = \bar{U} - U\left(x\right) \overset{!}{=} 0$

# Applying the Lagrange method

## Comparing the Lagrange multipliers

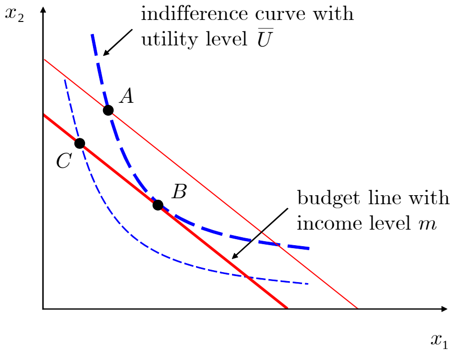
- $\lambda$ – the shadow price for utility maximization (translates additional income $m$ into higher utility: $\lambda = \frac{\partial V}{\partial m}$);
- $\mu$ – the shadow price for expenditure minimization (translates additional utility $\bar{U}$ into higher expenditure: $\mu = \frac{\partial e(p,\bar{U})}{\partial \bar{U}}$).

We note without proof

$$\mu = \frac{1}{\lambda}$$

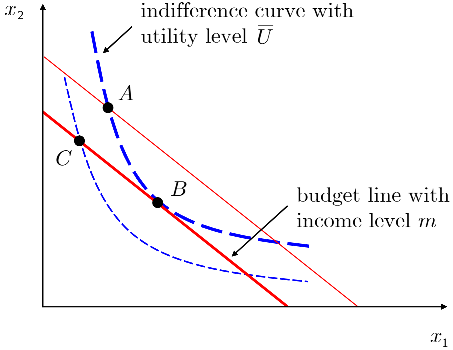
# The duality theorem

## here, duality does work




- At the budget line through point $C$ (budget $m$), the household optimum is at point $B$ and the utility is $U(B) = V(p, m)$.
- The expenditure needed to obtain $B$'s utility level or $A$'s utility level is equal to $m$:

$$e(p, V(p, m)) = m.$$




- The minimal expenditure for the indifference curve passing through $A$ (utility level $\bar{U}$) is denoted by $e\left(p,\bar{U}\right)$ and achieved by bundle $B$.
- With income $e\left(p,\bar{U}\right)$ (at point $C$), the highest achievable utility is $V\left(p,e\left(p,\bar{U}\right)\right)=\bar{U}$.

## Theorem

*Let $U : \mathbb{R}^{\ell}_{+} \to \mathbb{R}$ be a continuous utility function that obeys local nonsatiation and let $p >> 0$ be a price vector.* $\Rightarrow$

- *If $x(p, m)$ is the household optimum for $m > 0$:*

$$\chi(p, V(p, m)) = x(p, m)$$

$$e(p, V(p, m)) = m.$$

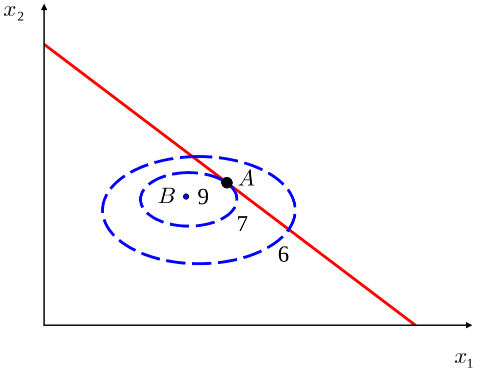
- *If $\chi(p, \bar{U})$ is the expenditure-minimizing bundle for $\bar{U} > U(0)$:*

$$x(p, e(p, \bar{U})) = \chi(p, \bar{U})$$

$$V(p, e(p, \bar{U})) = \bar{U}.$$

# The duality theorem

## here, duality does not work




- At $(p, m)$, the household optimum is at the bliss point with $V(p, m) = 9$.
- The expenditure needed to obtain this utility level is smaller than $m$:

$$e(p, V(p, m)) < m.$$

Here, local nonsatiation is violated.

# Main results

## Theorem

*Consider a household with a continuous utility function $U$, Hicksian demand function $\chi$ and expenditure function $e$.*

- *Shephard's lemma: The price increase of good $g$ by one small unit increases the expenditure necessary to uphold the utility level by $\chi_g$.*
- *Roy's identity: A price increase of good $g$ by one small unit decreases the budget available for the other goods by $\chi_g$ and indirect utility by the product of the marginal utility of income $\frac{\partial V}{\partial m}$ and $\chi_g$.*
- *Hicksian law of demand: If the price of a good $g$ increases, the Hicksian demand $\chi_g$ does not increase.*
- *The Hicksian cross demands are symmetric:* $\frac{\partial \chi_g(p,\bar{U})}{\partial p_k} = \frac{\partial \chi_k(p,\bar{U})}{\partial p_g}$.
- *Slutsky equations*
  *see next slide*

# Main results

## Theorem

- *Money-budget Slutsky equation:*

$$\frac{\partial x_g}{\partial p_g} = \frac{\partial \chi_g}{\partial p_g} - \frac{\partial x_g}{\partial m}\chi_g.$$

- *Endowment Slutsky equation:*

$$\frac{\partial x_g^{endowment}}{\partial p_g} = \frac{\partial \chi_g}{\partial p_g} + \frac{\partial x_g^{money}}{\partial m}\left(\omega_g - \chi_g\right).$$

# Shephard's lemma

- Assume a price increase for a good $g$ by one small unit.
- To achieve the same utility level, expenditure must be increased by at most

$$\frac{\partial e}{\partial p_g} \leq \chi_g$$

- Shephard's Lemma (see manuscript):

$$\frac{\partial e}{\partial p_g} = \chi_g$$

# Roy's identity

Duality equation:

$$\bar{U} = V\left(p, e\left(p, \bar{U}\right)\right).$$

Differentiating with respect to $p_g$:

$$\begin{aligned} 0 &= \frac{\partial V}{\partial p_g} + \frac{\partial V}{\partial m}\frac{\partial e}{\partial p_g} \\ &= \frac{\partial V}{\partial p_g} + \frac{\partial V}{\partial m}\chi_g \text{ (Shephard's lemma)} \end{aligned}$$

Roy's identity: $\frac{\partial V}{\partial p_g} = \frac{\partial V}{\partial m}\left(-\chi_g\right).$

# Compensated (Hicksian) law of demand

Assume:

- $p$ and $p'$ – price vectors from $\mathbb{R}^\ell$;
- $\chi(p, \bar{U}) \in \mathbb{R}^\ell_+$ and $\chi(p', \bar{U}) \in \mathbb{R}^\ell_+$ – expenditure-minimizing bundles necessary to achieve a utility of at least $\bar{U}$.

$$p' \cdot \underbrace{\chi(\bar{U}, p')}_{\substack{\text{expenditure-}\\\text{minimizing}\\\text{bundle at } p'}} \leq p' \cdot \underbrace{\chi(\bar{U}, p)}_{\substack{\text{expenditure-}\\\text{minimizing}\\\text{bundle at } p}} .$$

... (see the manuscript)

$\Rightarrow$ If the price of one good increases, the Hicksian demand for that good cannot increase: $\frac{\partial \chi_g}{\partial p_g} \leq 0$.

## Definition

Let $f : M \to \mathbb{R}$ be a function on a convex domain $M \subseteq \mathbb{R}^\ell$.

$\Rightarrow$

- $f$ is concave if

$$f\left(kx + (1-k)\,y\right) \geq kf\left(x\right) + (1-k)\,f\left(y\right)$$

  for all $x, y \in M$ and for all $k \in [0,1]$ (for $\leq$ – convex).

- $f$ is strictly concave if

$$f\left(kx + (1-k)\,y\right) > kf\left(x\right) + (1-k)\,f\left(y\right)$$

  holds for all $x, y \in M$ with $x \neq y$ and for all $k \in (0,1)$ (for $<$ – strictly convex).

# Hesse matrix

## Definition

Let $f : \mathbb{R}^\ell \to \mathbb{R}$ be a function.
The second-order partial derivative of $f$ with respect to $x_i$ and $x_j$ (if it exists) is given by

$$f_{ij}(x) := \frac{\partial \frac{\partial f(x)}{\partial x_i}}{\partial x_j} = \frac{\partial^2 f(x)}{\partial x_i \partial x_j}.$$

If all the second-order partial derivatives exist, the Hesse matrix of $f$ is given by

$$H_f(x) = \begin{pmatrix} f_{11}(x) & f_{12}(x) & & f_{1\ell}(x) \\ f_{21}(x) & f_{22}(x) & & \\ & & & \\ f_{\ell 1}(x) & f_{n2}(x) & & f_{\ell\ell}(x) \end{pmatrix}.$$

## Problem

*Determine the Hesse matrix for* $f(x, y) = x^2 y + y^2$.

# Hesse matrix

Believe me (or check in the script) that

## Lemma (diagonal entries)

**If** *a function* $f : \mathbb{R}^\ell \to \mathbb{R}$ *is concave (strictly concave), the diagonal entries of its Hesse matrix are non-positive (negative).*

Remember: For $f : \mathbb{R} \to \mathbb{R}$, $f$ is concave **iff** $f''(x) \leq 0$ for all $x \in \mathbb{R}$ (chapter on von Neumann-Morgenstern utility)

## Lemma (symmetry)

*If all the second-order partial derivatives of* $f : \mathbb{R}^\ell \to \mathbb{R}$ *exist and are continuous, then*

$$f_{ij}(x) = f_{ji}(x) \text{ for all } i, j = 1, ..., \ell$$

## Lemma (expenditure fct. concave)

*The expenditure function is concave.*

# Hesse matrix of expenditure function

$$H_e\left(p,\bar{U}\right)=\begin{pmatrix} \frac{\partial^2 e(p,\bar{U})}{(\partial p_1)^2} & \frac{\partial^2 e(p,\bar{U})}{\partial p_1 \partial p_2} & & \frac{\partial^2 e(p,\bar{U})}{\partial p_1 \partial p_\ell} \\ \frac{\partial^2 e(p,\bar{U})}{\partial p_2 \partial p_1} & \frac{\partial^2 e(p,\bar{U})}{(\partial p_2)^2} & & \\ & & & \\ \frac{\partial^2 e(p,\bar{U})}{\partial p_\ell \partial p_1} & \frac{\partial^2 e(p,\bar{U})}{\partial p_\ell \partial p_2} & & \frac{\partial^2 e(p,\bar{U})}{(\partial p_\ell)^2} \end{pmatrix}.$$

# Compensated (Hicksian) law of demand

By Shephard's lemma:

$$\frac{\partial e\left(p, \bar{U}\right)}{\partial p_g} = \chi_g\left(p, \bar{U}\right)$$

Forming the derivative of the Hicksian demand, we find

$$\frac{\partial \chi_g\left(p, \bar{U}\right)}{\partial p_k} = \frac{\partial \frac{\partial e(p, \bar{U})}{\partial p_g}}{\partial p_k} = \frac{\partial^2 e\left(p, \bar{U}\right)}{\partial p_g \partial p_k}$$

with two interesting conclusions:

1. $g = k$
   Hicksian law of demand (by lemma "expenditure fct. concave" and lemma "diagonal entries"):

$$\frac{\partial \chi_g\left(p, \bar{U}\right)}{\partial p_g} \leq 0$$

# Substitutes and complements (the Hicksian definition)

2. $g \neq k$ :
   If the off-diagonal entries in the expenditure function's Hesse matrix are continuous, lemma "symmetry" implies

$$\frac{\partial \chi_g\left(p, \bar{U}\right)}{\partial p_k} = \frac{\partial \chi_k\left(p, \bar{U}\right)}{\partial p_g}.$$

## Definition

Goods $g$ and $k$ are

- substitutes if

$$\frac{\partial \chi_g\left(p, \bar{U}\right)}{\partial p_k} = \frac{\partial \chi_k\left(p, \bar{U}\right)}{\partial p_g} \geq 0;$$

- complements if

$$\frac{\partial \chi_g\left(p, \bar{U}\right)}{\partial p_k} = \frac{\partial \chi_k\left(p, \bar{U}\right)}{\partial p_g} \leq 0.$$

# Every good has at least one substitute.

For all $\alpha > 0$ :

$$\chi_g\left(\alpha p, \bar{U}\right) = \chi_g\left(p, \bar{U}\right).$$

Differentiating with the adding rule (chapter on preferences) yields

$$\sum_{k=1}^{\ell} \frac{\partial \chi_g\left(\alpha p, \bar{U}\right)}{\partial p_k} \cdot p_k = \frac{\partial \chi_g\left(\alpha p, \bar{U}\right)}{\partial \alpha} = \frac{\partial \chi_g\left(p, \bar{U}\right)}{\partial \alpha} = 0$$

$\Rightarrow$

$$\sum_{\substack{k=1,\\ k\neq g}}^{\ell} \frac{\partial \chi_g\left(\alpha p, \bar{U}\right)}{\partial p_k} \cdot p_k = -\underbrace{\frac{\partial \chi_g\left(\alpha p, \bar{U}\right)}{\partial p_g}}_{\leq 0} \cdot p_g \geq 0.$$

$\Rightarrow$

**Lemma**

*Assume $\ell \geq 2$ and $p >> 0$. Every good has at least one substitute.*

# Three effects of a price increase

$$\frac{\partial \chi_1\left(p, \bar{U}\right)}{\partial p_1} \underset{!}{\leq} 0 \text{ and } \frac{\partial x_1\left(p, m\right)}{\partial p_1} \underset{?}{\leq} 0$$

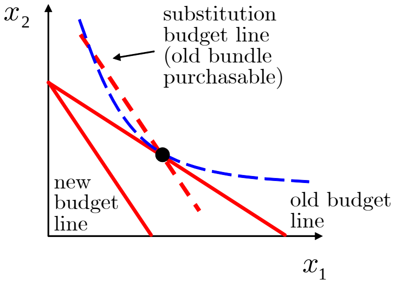
1. Substitution effect or opportunity-cost effect: $p_1 \uparrow$
   - $\Rightarrow p_1 / p_2 \uparrow$
   - $\Rightarrow x_1 \downarrow$ and $x_2 \uparrow$
2. Consumption-income effect: $p_1 \uparrow$
   - $\Rightarrow$ overall consumption possibilities decrease
   - $\Rightarrow x_1 \downarrow$ if 1 is a normal good
3. Endowment-income effect: $p_1 \uparrow$
   - $\Rightarrow$ value of endowment increases
   - $\Rightarrow x_1 \uparrow$ if 1 is a normal good

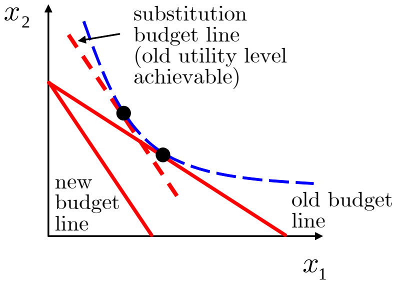
# Two different substitution effects

## Definitions

In response to a price change, there are two different ways to keep real income constant:

- Old-household-optimum substitution effect
- Old-utility-level substitution effect



# Two different substitution effects

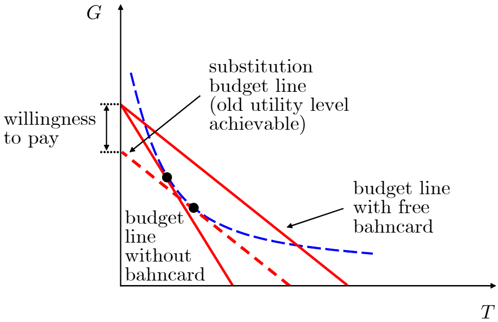
## Bahncard 50

### Example

Two goods: train rides $T$ and other goods $G$

- $p_T = 0.2$ (per kilometer), $p_G = 1$.
- "Bahncard 50": $p_T$ reduced to 0.1.

Willingness to pay for the "Bahncard 50"?



# Two different substitution effects

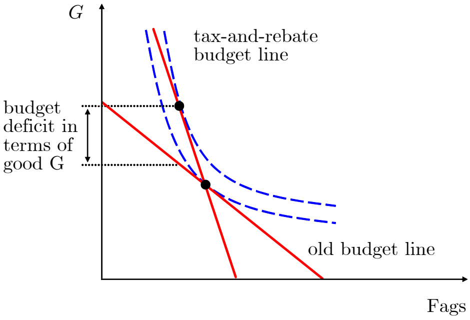
## Tax and rebate

**Example**

You smoke 10 cigarettes per day. The government is concerned about your health.

- Quantity tax of 10 cents, but
- rebate of 1 Euro per day.

Budget deficit in terms of the other goods?



# The Slutsky equation for the money budget

## Derivation

Duality equation: $\chi_g(p, \bar{U}) = x_g(p, e(p, \bar{U}))$
Differentiate with respect to $p_k$

$$\begin{aligned}\frac{\partial \chi_g}{\partial p_k} &= \frac{\partial x_g}{\partial p_k} + \frac{\partial x_g}{\partial m}\frac{\partial e}{\partial p_k} \\ &= \frac{\partial x_g}{\partial p_k} + \frac{\partial x_g}{\partial m}\chi_k \text{ (Shephard's Lemma)}\end{aligned}$$

The Slutsky equation $(g = k)$:

$$\frac{\partial x_g}{\partial p_g} = \underbrace{\frac{\partial \chi_g}{\partial p_g}}_{\substack{\leq 0 \\ \text{Hicksian} \\ \text{law of demand}}} - \underbrace{\frac{\partial x_g}{\partial m}}_{\substack{> 0 \\ \text{for normal goods}}} \chi_g.$$

# The Slutsky equation for the money budget

## Implications

The Slutsky equation:

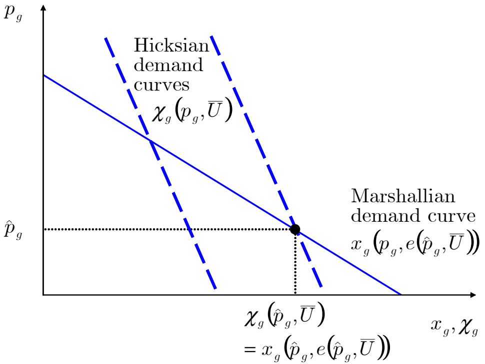
$$\frac{\partial x_g}{\partial p_g} = \underbrace{\frac{\partial \chi_g}{\partial p_g}}_{\substack{\leq 0 \\ \text{Hicksian} \\ \text{law of demand}}} - \underbrace{\frac{\partial x_g}{\partial m}}_{\substack{> 0 \\ \text{for normal goods}}} \chi_g .$$

- $g$ normal $\Rightarrow$ $g$ ordinary
- $g$ normal $\Rightarrow$ effect of a price increase stronger on Marshallian demand than on Hicksian demand
- $g$ inferior $\Rightarrow$ income effect may outweigh substitution effect —> Giffen good

# The Slutsky equation for the money budget

Assume $(\hat{p}_g, \bar{U})$.

- By duality, $\chi_g(\hat{p}_g, \bar{U}) = x_g(\hat{p}_g, e(\hat{p}_g, \bar{U}))$.
- $g$ normal $\Rightarrow$ Hicksian demand curves steeper than Marshallian demand curves



$$
\begin{aligned}
& \frac{\partial x_g^{\text{endowment}}(p,\omega)}{\partial p_k} \\
= & \frac{\partial x_g^{\text{money}}(p, p \cdot \omega)}{\partial p_k} \\
= & \frac{\partial x_g^{\text{money}}}{\partial p_k} + \frac{\partial x_g^{\text{money}}}{\partial m}\frac{\partial (p \cdot \omega)}{\partial p_k} \\
= & \frac{\partial x_g^{\text{money}}}{\partial p_k} + \frac{\partial x_g^{\text{money}}}{\partial m}\omega_k \text{ (definition of dot product)} \\
= & \left(\frac{\partial \chi_g}{\partial p_k} - \frac{\partial x_g^{\text{money}}}{\partial m}\chi_k\right) + \frac{\partial x_g^{\text{money}}}{\partial m}\omega_k \text{ (money-budget Slutsky equation)} \\
= & \frac{\partial \chi_g}{\partial p_k} + \frac{\partial x_g^{\text{money}}}{\partial m}(\omega_k - \chi_k).
\end{aligned}
$$

# The Slutsky equation for the endowment budget

## Implications

The Slutsky equation:

$$\frac{\partial x_g^{\text{endowment}}}{\partial p_g} = \underbrace{\frac{\partial \chi_g}{\partial p_g}}_{\leq 0} + \underbrace{\frac{\partial x_g^{\text{money}}}{\partial m}}_{\substack{>0 \\ \text{for a normal} \\ \text{good } g}} \underbrace{(\omega_g - \chi_g)}_{\substack{<0 \\ \text{for net demander}}} .$$

- $g$ normal and household net demander $\Rightarrow$ $g$ ordinary
- $g$ normal and household net supplier $\Rightarrow$ $g$ may be non-ordinary

# The Slutsky equation for the endowment budget

## Application: consumption today versus consumption tomorrow

- The intertemporal budget equation in future value terms:

$$(1+r)\,x_1 + x_2 = (1+r)\,\omega_1 + \omega_2.$$

- The Slutsky equation:

$$\frac{\partial x_1^{\text{endowment}}}{\partial(1+r)} = \underbrace{\frac{\partial \chi_1}{\partial(1+r)}}_{\leq 0} + \underbrace{\frac{\partial x_1^{\text{money}}}{\partial m}}_{\substack{>0 \\ \text{for normal good} \\ \text{first-period consumption}}} \underbrace{(\omega_1 - \chi_1)}_{\substack{>0 \\ \text{for lender}}}.$$

# The Slutsky equation for the endowment budget

## Application: leisure versus consumption

- The budget equation:

$$wx_R + px_C = w24 + p\omega_C.$$

- The Slutsky equation:

$$\frac{\partial x_R^{\text{endowment}}}{\partial w} = \underbrace{\frac{\partial \chi_R}{\partial w}}_{\leq 0} + \underbrace{\frac{\partial x_R^{\text{money}}}{\partial m}}_{\substack{>0 \\ \text{for normal} \\ \text{good recreation}}} \underbrace{(24 - \chi_R)}_{\substack{\geq 0 \\ \text{by definition}}}.$$

Thus, if the wage rate increases, it may well happen that the household works ...

- The budget equation:

$$\frac{\gamma}{1-\gamma}x_1 + x_2 = \frac{\gamma}{1-\gamma}(A-D) + A$$

with $\gamma K$ – payment to the insurance if $K$ is to be paid to the insuree in case of damage $D$.

- The Slutsky equation for consumption in case of damage:

$$\frac{\partial x_1^{\text{endowment}}}{\partial \frac{\gamma}{1-\gamma}} = \underbrace{\frac{\partial \chi_1}{\partial \frac{\gamma}{1-\gamma}}}_{\leq 0} + \underbrace{\frac{\partial x_1^{\text{money}}}{\partial m}}_{\substack{>0 \\ \text{for normal} \\ \text{good consumption} \\ \text{in case of damage}}} \underbrace{(A-D-\chi_1)}_{\substack{\leq 0 \\ \text{in case of} \\ \text{a nonnegative insurance}}} .$$

1. The duality approach
2. Shephard's lemma
3. The Hicksian law of demand
4. Slutsky equations
5. **Compensating and equivalent variations**

# Compensating and equivalent variations

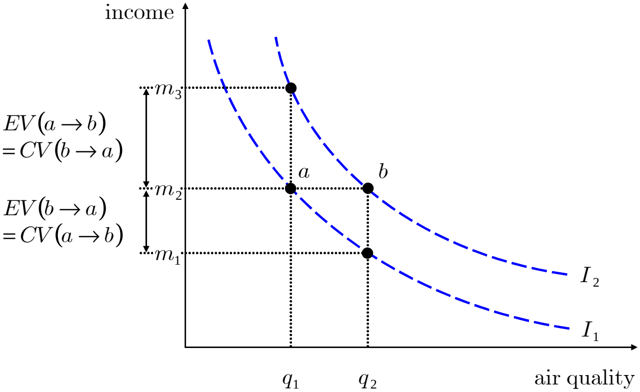
## Definition

- A variation is equivalent to an event, if both (the event or the variation) lead to the same indifference curve $\rightarrow EV\,(\text{event})$;
- A variation is compensating if it restores the individual to its old indifference curve (prior to the event) $\rightarrow CV\,(\text{event})$.

# Compensating and equivalent variations

| **Equivalent variation** | **Compensating variation** |
|---|---|
| **in lieu of an event**<br><br>monetary variation<br>is equivalent<br>(i.e., achieving the same utility) | **because of an event**<br><br>monetary variation<br>compensates for event<br>(i.e., holding utility constant) |

Change of air quality:



# Compensating and equivalent variations

- Compensation money $\rightarrow$ if some amount of money is given to the individual:
  $CV$ (degr.) – the compensation money for the degradation of the air quality.
- Willingness to pay $\rightarrow$ if money is taken from the individual.
  $EV$ (degr.) – the willingness to pay for the prevention of the degradation.
- If the variation turns out to be negative, exchange $-EV$ for $EV$ or $EV$ for $-EV$ (similarly for $CV$).

# Compensating or equivalent variation?

## Example

- Consumer's compensating variation: A consumer asks himself how much he is prepared to pay for a good.
- Consumer's equivalent variation – the compensation payment for not getting the good. You go into a shop and ask for compensation for not taking (stealing?) the good.
- Producer's compensating variation – the compensation money he gets for selling a good.
- Producer's equivalent variation: The producer asks himself how much he would be willing to pay if the good were not taken away from him.

# Price changes

- The willingness to pay for the price decrease of good $g$:

$$CV\left(p_g^h \to p_g^l\right) = EV\left(p_g^l \to p_g^h\right).$$

- The compensation money for the price increase of good $g$:

$$EV\left(p_g^h \to p_g^l\right) = CV\left(p_g^l \to p_g^h\right).$$

- $CV\left(p_1^h \to p_1^l\right) < EV\left(p_1^h \to p_1^l\right)$ (for normal goods, see below);
- $cv$ and $ev$ – if we are not sure whether a change is good or bad.

**Lemma**

*Consider the event of a price change from $p^{old}$ to $p^{new}$. Then:*

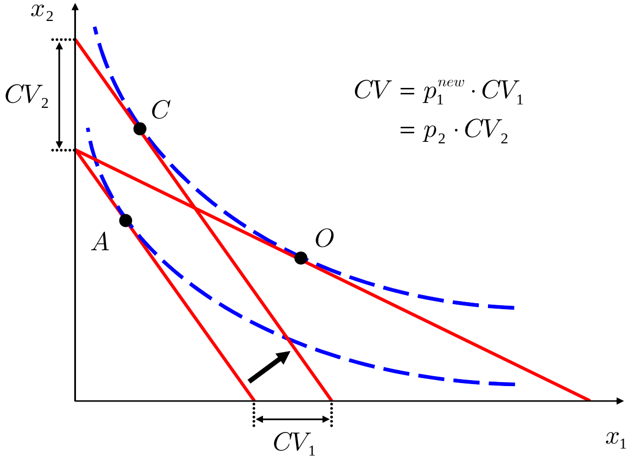
$$\begin{aligned} U^{old} &:= V\left(p^{old}, m\right) = V\left(p^{new}, m + cv\right),\ CV = |cv| \text{ and} \\ U^{new} &:= V\left(p^{new}, m\right) = V\left(p^{old}, m + ev\right), EV = |ev|. \end{aligned}$$

## Problem

*Tell the sign of $cv$ and $ev$ for a price increase of all goods.*

# Price changes

Price increase of good 1:




$$CV = p_1^{new} \cdot CV_1$$
$$= p_2 \cdot CV_2$$

# Price changes

## Example

Cobb-Douglas utility function: $u(x_1, x_2) = x_1^a x_2^{1-a}$ with $(0 < a < 1)$.
By a price decrease from $p_1^h$ to $p_1^l < p_1^h$ (for example, Bahncard 50)

$$\underbrace{\left(a\frac{m}{p_1^h}\right)^a \left((1-a)\frac{m}{p_2}\right)^{1-a}}_{\text{utility at the old, high price}}$$

$$= \underbrace{\left(a\frac{m + cv\left(p_1^h \to p_1^l\right)}{p_1^l}\right)^a \left((1-a)\frac{m + cv\left(p_1^h \to p_1^l\right)}{p_2}\right)^{1-a}}_{\text{utility at the new, lower price and compensating variation}}.$$

$$cv\left(p_1^h \to p_1^l\right) = -\left(1 - \left(\frac{p_1^l}{p_1^h}\right)^a\right) m < 0.$$

## Problem

*Determine the equivalent variation for a price decrease in case of Cobb-Douglas utility preferences.*

## Problem

*Determine the compensating variation and the equivalent variation for the price decrease from $p_1^h$ to $p_1^l < p_1^h$ and the quasi-linear utility function given by*

$$u(x_1, x_2) = \ln x_1 + x_2 \qquad (x_1 > 0)!$$

*Assume $\frac{m}{p_2} > 1$! Hint: the household optimum is $x(m, p) = \left(\frac{p_2}{p_1}, \frac{m}{p_2} - 1\right)$.*

# Applying duality

## Implicit definition of compensating variation

Implicit definition: $U^{old} := V\left(p^{old}, m\right) = V\left(p^{new}, m + cv\right)$
Duality equation $e\left(p, V\left(p, m\right)\right) = m$ leads to

$$\begin{aligned} e\left(p^{old}, V\left(p^{old}, m\right)\right) &= m && (1) \\ e\left(p^{new}, V\left(p^{new}, m + cv\right)\right) &= m + cv && (2) \end{aligned}$$

$\Rightarrow$

$$\begin{aligned} cv &= e\left(p^{new}, V\left(p^{new}, m + cv\right)\right) - m && (2) \\ &= e\left(p^{new}, U^{old}\right) - e\left(p^{old}, U^{old}\right) && \text{(1) and implicit definition} \end{aligned}$$

The household is given, or is relieved of, the money necessary to uphold the old utility level.

Implicit definition: $U^{new} := V\left(p^{new}, m\right) = V\left(p^{old}, m + ev\right)$
Duality equation $e\left(p, V\left(p, m\right)\right) = m$ leads to

$$
\begin{aligned}
e\left(p^{new}, V\left(p^{new}, m\right)\right) &= m & (1)\\
e\left(p^{old}, V\left(p^{old}, m + ev\right)\right) &= m + ev & (2)
\end{aligned}
$$

$\Rightarrow$

$$
\begin{aligned}
ev &= e\left(p^{old}, V\left(p^{old}, m + ev\right)\right) - m && (2)\\
&= e\left(p^{old}, U^{new}\right) - e\left(p^{new}, U^{new}\right) && (1) \text{ and implicit definition}
\end{aligned}
$$

Assume $p^{new} < p^{old}$. The equivalent variation is the amount of money necessary to increase the household's income from $m = e\left(p^{new}, U^{new}\right)$ to $e\left(p^{old}, U^{new}\right)$.

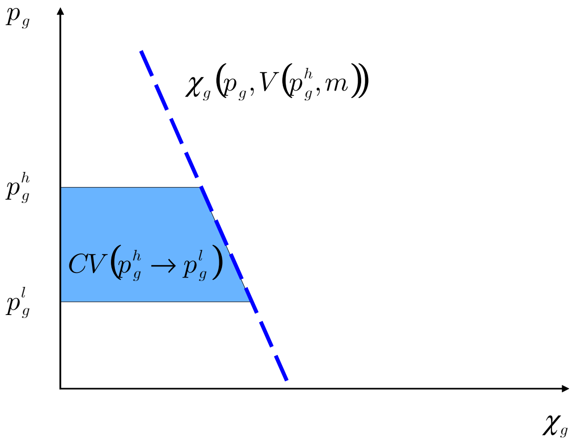
$$cv\left(p_g^h \to p_g^l\right) = -\int_{p_g^l}^{p_g^h} \chi_g\left(p_g, V\left(p_g^h, m\right)\right) dp_g$$

by (if you want)

$$\begin{aligned}
cv\left(p_g^h \to p_g^l\right) &= e\left(p_g^l, V\left(p_g^h, m\right)\right) - e\left(p_g^h, V\left(p_g^h, m\right)\right) \\
&= -\left[e\left(p_g^h, V\left(p_g^h, m\right)\right) - e\left(p_g^l, V\left(p_g^h, m\right)\right)\right] \\
&= -\, e\left(p_g, V\left(p_g^h, m\right)\right)\Big|_{p_g^l}^{p_g^h} \\
&= -\int_{p_g^l}^{p_g^h} \frac{\partial e\left(p, V\left(p_g^h, m\right)\right)}{\partial p_g} dp_g \text{ (Fundamental Theorem)} \\
&= -\int_{p_g^l}^{p_g^h} \chi_g\left(p_g, V\left(p_g^h, m\right)\right) dp_g \text{ (Shephard's lemma).}
\end{aligned}$$

# Variations for a price change and Hicksian demand

## Applying the fundamental theorem of calculus



## Theorem

*Assume any good $g$ and any price decrease from $p_g^h$ to $p_g^l < p_g^h$.*

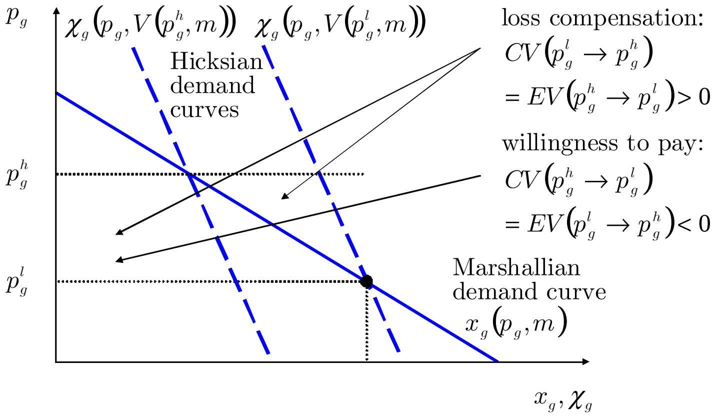
$$cv\left(p_g^h \to p_g^l\right) = -\int_{p_g^l}^{p_g^h} \chi_g\left(p_g, V\left(p_g^h, m\right)\right) dp_g.$$

*If $g$ is a normal good:*

$$\underbrace{CV\left(p_g^h \to p_g^l\right)}_{\substack{\text{(Hicksian)} \\ \text{willingness to pay}}} \leq \underbrace{\int_{p_g^l}^{p_g^h} x_g\left(p_g\right) dp_g}_{\substack{\text{Marshallian} \\ \text{willingness to pay}}} \leq \underbrace{CV\left(p_g^l \to p_g^h\right)}_{\substack{\text{(Hicksian)} \\ \text{loss compensation}}}.$$

# Variations for a price change and Hicksian demand

## Comparisons for normal goods



## Definition

The Hicksian consumer's rent at price $\hat{p}_g < p_g^{proh}$ is given by

$$\begin{aligned} CR^{Hicks}\left(\hat{p}_g\right) \quad : \quad &= CV\left(p_g^{proh} \rightarrow \hat{p}_g\right) \\ &= \int_{\hat{p}_g}^{p_g^{proh}} \chi_g\left(p_g, V\left(p_g^{proh}, m\right)\right) dp_g. \end{aligned}$$

# Further exercises I

Problem 1
Determine the expenditure functions and the Hicksian demand function for $U(x_1, x_2) = \min(x_1, x_2)$ and $U(x_1, x_2) = 2x_1 + x_2$. Can you confirm the duality equations

$$\begin{aligned} \chi(p, V(p, m)) &= x(p, m) \text{ and} \\ x(p, e(p, \bar{U})) &= \chi(p, \bar{U})? \end{aligned}$$

# Further exercises II

Problem 2
Derive the Hicksian demand functions and the expenditure functions of the two utility functions:

(a) $U(x_1, x_2) = x_1 \cdot x_2$,
(b) $U(x_1, x_2) = \min(a \cdot x_1, b \cdot x_2)$ with $a, b > 0$.

Problem 3
Verify Roy's identity for the utility function $U(x_1, x_2) = x_1 \cdot x_2$!

Problem 4
Draw a figure that shows the equivalent variation following a price increase.